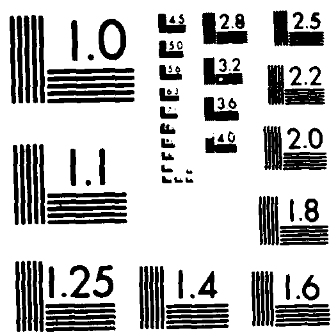
AD-A193 477 A LIMITING 'VISCOSITY' APPROACH TO THE RIEMANN PROBLEM FOR MATERIALS EXHI..(U) WISCONSIN UNIV-MADISON CENTER FOR MATHEMATICAL SCIENCES M SLEMROD SEP 87 1/1

UNCLASSIFIED CMS-TSR-88-6 AFOSR-85-0239 F/G 20/4 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963 A

CMS Technical Summary Report #88-6

A LIMITING "VISCOSITY" APPROACH TO THE RIEMANN PROBLEM FOR MATERIALS EXHIBITING CHANGE OF PHASE

M. Slemrod

UNIVERSITY OF WISCONSIN

UNIVERSITY OF WISCONSIN MADISON

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

September 1987

(Received September 28, 1987)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Air Force Office of Scientific Research
Building 410
Bolling AFB
Washington, DC 20332

- A -

UNIVERSITY OF WISCONSIN-MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

A LIMITING "VISCOSITY" APPROACH TO THE RIEMANN PROBLEM FOR MATERIALS EXHIBITING CHANGE OF PHASE

M. Slemrod[1,2]

Technical Summary Report #88-6
September 1987

ABSTRACT

This paper considers resolution of the Riemann problem for a van der Waals fluid. The method of an analysis is one of limiting artificial viscosity. (Keywords: connecting orbits, a prior estimates, theorems),

AMS (MOS) Subject Classifications: 35M05, 76T05

Key Words: conservation laws, mixed hyperbolic-elliptic system, phase transitions

---

[1]Erna and Jakob Michael Visiting Professor, Weizmann Institute of Science, Rehovot, Israel.

[2]This research was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Contract/Grant No. AFOSR-85-0239 (R.P.I.) and by the United States Army, Army Research Office under Contracts/Grants No. 5-28529 and No. 5-28317 (Brown University). The United States Government is authorized to reproduce and distribute reprints for Government purposes not withstanding any copyright herein.

A Limiting "Viscosity" Approach to the Riemann Problem for Materials Exhibiting Change of Phase

M. Slemrod[1,2]
Center for the Mathematical Sciences
University of Wisconsin
Madison, WI 53706, USA

0. Introduction

The one dimensional isothermal motion of a compressible elastic fluid or solid can be described in Lagrangian coordinates by the coupled system

$$u_t + p(w)_x = 0, \tag{0.1}$$

$$w_t - u_x = 0. \tag{0.2}$$

Here u denotes the velocity, w the specific volume for a fluid (or displacement gradient for a solid), and $-p$ is the stress which must be determined through a constitutive relation to w. For many materials a natural condition placed on p is that $p'(w) < 0$ for all values of w (or all positive values of w) depending on the context of the problem. This makes (0.1), (0.2) a coupled system of hyperbolic conservation laws. In this paper, however, we shall consider the case where p has a graph illustrated by Figure 1. For convenience p will be globally defined, smooth, with

$$p' < 0 \qquad w < \alpha, \qquad w > \beta; \qquad p' > 0, \qquad \alpha < w < \beta;$$

$$p''(\alpha) > 0, \qquad p''(\beta) < 0.$$

---

[1] Erna and Jakob Michael Visiting Professor, Weizmann Institute of Science, Rehovot, Israel.

[2] This research was supported in part by the Air force Office of Scientific Research, Air Force Systems Command, USAF, under Contract/Grant No. AFOSR-85-0239 (R.P.I.) by the United States Army, Army Research Office under Contracts/Grants Nos. 5-28529 and 5-28317 (Brown University). The United States Government is authorized to reproduce and distribute reprints for Government purposes not withstanding any copyright herein.

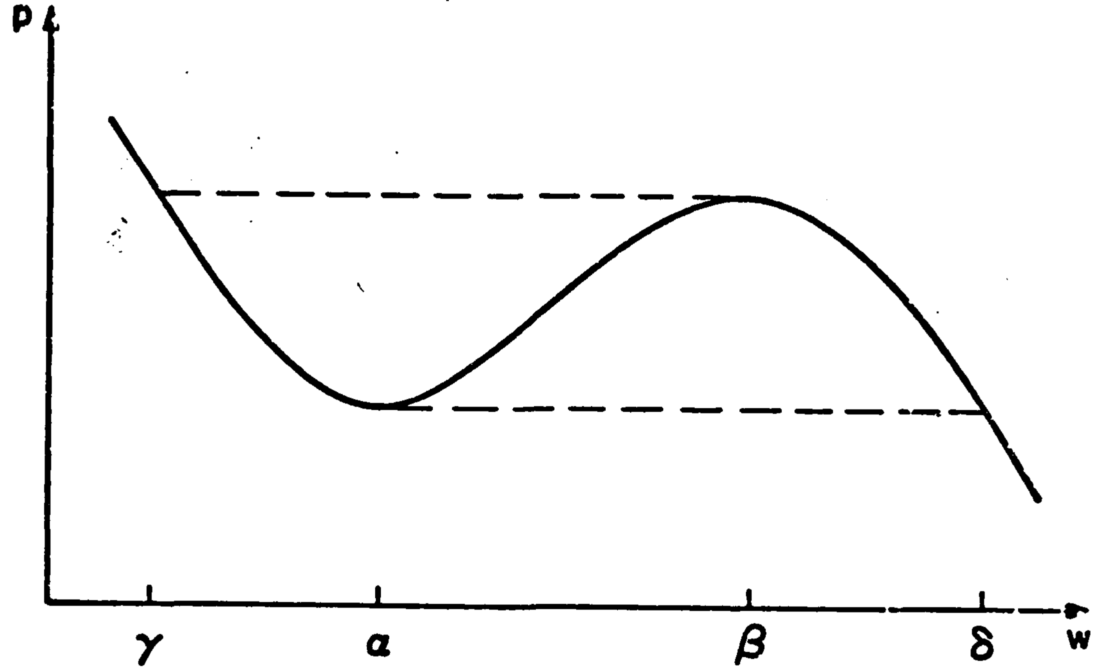


Figure 1.

This type of constitutive relation is usually associated with a van der Waals fluid where

$$p(w) = \frac{RT}{w-b} - \frac{a}{w^2}$$

and R, a, b are positive material constants, T is the temperature. Here we need nothing as specific as the van der Waals constitutive relation though our results will strongly depend at times on the global behavior of p as $|w| \to \infty$.

The reason for this non-standard choice of p is that it serves as a prototype problem for the dynamics of a material exhibiting changes of phase. For example in a van der Waals fluid the states $w < \alpha$ are viewed as liquid while states with $w > \beta$ are viewed as vapor. The non-monotonicity of p allows co-existence of liquid and vapor phases.

The evolution of (0.1), (0.2) will be governed by initial data. Here we pose piecewise constant data

$$u(x,0) = \begin{matrix} u_- \\ u_+ \end{matrix} \qquad w(x,0) = \begin{matrix} w_- & x < 0 \\ w_+ & x \geq 0 \end{matrix} \tag{0.3}$$

which makes (0.1)-(0.3) into a *mixed hyperbolic-elliptic Riemann initial value problem* which we call *problem P*.

The classic method of solution of the Riemann problem is based on the construction of shock and wave curves for (0.1), (0.2). For van der Waals like materials discussions of this approach have been given first by R. James [1] and later by M. Shearer [2], [3]. The difficulty with this procedure is that even if shock admissibility conditions are known a priori it is not obvious in what manner the full solution which is a composite of shock and rarefaction waves is admissible. For example in [12], Shearer proves existence of solutions to P when the data w_-, w_+ are in different phases but close to the well known Maxwell line. Each discontinuity in Shearer's solution *is* admissible with respect to the viscosity-capillarity criterion discussed below.

The investigation here is based on a different approach. First we recall that earlier work [4], [5], [6] has suggested a reasonable admissibility criteria for (0.1), (0.2) to be the following *viscosity-capillarity criterion.* Namely weak solutions of (0.1), (0.2) will be admissible if they are limits boundedly a.e. of solutions u_ν, w_ν of the system.

$$u_t + p(w)_x = \nu u_{xx} - \nu^2 A w_{xxx}, \tag{0.4}$$

$$w_t - u_x = 0, \tag{0.5}$$

as $\nu \to 0+$.

This system is derived from Korteweg's theory of capillarity where the total stress is written as the sum $-p(w) + \nu u_x - \nu^2 A w_{xx}$ comprising elastic, viscous, and capillarity contributions.

As noted in [6] the substitutions

$$\begin{matrix} D_1 \\ D_2 \end{matrix} = \frac{\nu}{2} \mp \frac{\nu}{2}(1-4A)^{1/2}, \quad v = u - D_2 w_x \quad \text{bring} \qquad ((0.4),(0.5))$$

into the parabolic form

$$v_t + p(w)_x = D_1 v_{xx}, \tag{0.6}$$

$$w_t - v_x = D_2 w_{xx}, \tag{0.7}$$

when $0 \le A \le 1/4$. In particular the choice of $A = 1/4$, $\nu = 2\epsilon$ would say that admissible solutions would be limits as $\epsilon \to 0+$ of solutions of (0.6), (0.7) with $D_1 = D_2 = \epsilon$.

While (0.6), (0.7) is the common form for most artificial "viscosity" arguments, Kalasnikov (1959 [13]) Tupciev (1964 [14], 1972 [15]), Dafermos (1973 [7], 1974 [16]), and Dafermos and DiPerna (1976 [8]) suggested a variant advantageous for the study of Riemann initial value problems. Within the context of (0.1)-(0.2) the idea is to replace (0.6)-(0.7) with the system

$$u_t + p(w)_x = \epsilon t u_{xx}, \tag{0.8}$$

$$w_t - u_x = \epsilon t w_{xx}, \tag{0.9}$$

which is invariant under the transformation $(x,t) \to (ax, at)$, $a > 0$. (Here the letter v has been replaced by its former self u.) System (0.8), (0.9) has a decided advantage over (0.6), (0.7) in that it admits solutions that are functions of the single variable $\xi = \frac{x}{t}$. In fact a simple computation shows that $u(\frac{x}{t})$, $w(\frac{x}{t})$ is a solution of (0.8), (0.9), (0.3) if $u(\xi)$, $w(\xi)$ is a solution of the coupled system of non-autonomous ordinary differential equations

$$\epsilon u'' = p(w)' - \xi u', \tag{0.10}$$

$$\epsilon w'' = -u' - \xi w', \tag{0.11}$$

with boundary conditions

$$\begin{aligned} u(-\infty) &= u_-, \qquad & w(-\infty) &= w_- \\ u(+\infty) &= u_+ \qquad & w(+\infty) &= w_+ \end{aligned} . \tag{0.12}$$

Here $'$ denotes differentiation with respect to ξ. We will call *the boundary value problem (0.10)-(0.12) problem P_ϵ.*

Our program can now be broken up into two steps. The first part carried out in Sections 1 and 2 establishes that if the data are in different phases there is a solution of P_ϵ which exhibits one change of phase. Also we give special conditions on the one phase data which yield a one phase solution of P_ϵ. The main feature of the proofs of these results is to note that Dafermos's arguments in [7] (which provided the successful resolution of P_ϵ in the case $p' < 0$) and those of Dafermos and DiPerna in [8] do not directly apply. However a careful modification involving changes of underlying function spaces, application of the Leray-Schauder degree, and a new set of a priori estimates yield solvability of P_ϵ.

In Sections 3 and 4 we pursue the second part of our program, i.e. to give conditions on which solutions $u_\epsilon(\xi)$, $w_\epsilon(\xi)$ of P_ϵ possess limits as $\epsilon \to 0+$ which solve the Riemann problem P. In the case $p' < 0$, Dafermos [7] and Dafermos and DiPerna [8] succeeded via this method to solve P. Here we modify the ideas of [7], [8] to the case when $p' > 0$ in (α, β). In this case when the above mentioned special data are in the same phase assumptions on $p''(w)$ and behavior of p at infinity yield estimates on the total variation of u_ϵ, w_ϵ which combined with Helly's theorem shows u_ϵ, w_ϵ do converge to a solution of P. For data in different phases similar estimates in the total variation may be obtained to yield solvability of P except in one case. The case in doubt is when there is a sequence $\tau^\epsilon \to 0$ so that $|u_\epsilon(\tau^\epsilon)|$ becomes infinite as $\epsilon \to 0+$. For this case we know u_ϵ, w_ϵ possesses a subsequence which converges a.e. to function u, w as $\epsilon \to 0+$. The limit functions u, w will be a solution of the Riemann problem if and only if the pressure p equilibrates across the stagnant phase boundary $\xi = 0$, i.e., $\lim_{\xi \to 0+} p(w(\xi)) = \lim_{\xi \to 0-} p(w(\xi))$ (Theorem 4.13). Modulo this one case we see that the idea of artificial "viscosity" arguments which play such a vital role in the existence theory of hyperbolic conservation laws can be extended to mixed hyperbolic-elliptic systems as well. (In this regard see also [9] for a study of a viscosity approach to a mixed hyperbolic-elliptic *boundary* value problem.)

1. Existence of connecting orbits assuming a priori estimates

In this section we will establish an *existence theorem* for the connecting orbit problem P_ϵ described in the introduction under the assumption of a priori estimates on (u, w). With this goal in mind, consider the two-parameter family of problems

$$\epsilon u'' = \mu p(w)' - \xi u' \tag{1.1}$$

$$\epsilon w'' = \mu u' - \xi w' \tag{1.2}$$

$$u(-L) = u_-, \quad u(L) = u_+, \quad w(-L) = w_-, \quad w(L) = w_+, \tag{1.3}$$

where $\mu \in [0,1]$ and $L \geq 1$.

Theorem 1.1 Assume $w_- < \alpha$, $w_+ > \beta$ ($w_- > \beta, w_+ < \alpha$) and there is a constant M_0 such that every possible solution of (1.1), (1.2), (1.3) with $w'(\xi) > 0$ ($w'(\xi) < 0$) when $\alpha \leq w(\xi) \leq \beta$ satisfies the a priori estimate

$$\sup_{-L<\xi<L} \left(|u(\xi)| + |u'(\xi)| + |w(\xi)| + |w'(\xi)|\right) \leq M_0. \tag{H0}$$

Here M_0 can depend on $u_-, u_+, w_-, w_+, \epsilon, p$ but is independent of μ and L. Then their exist solutions of P_ϵ which satisfy the constraint that $w'(\xi) > 0$ ($w'(\xi) < 0$) if $\alpha \leq w(\xi) \leq \beta$, i.e. *the one phase change data connecting orbit problem possesses a one phase change solution.*

Proof. We consider the case $w_- < \alpha$, $w_+ > \beta$. The case $w_- > \beta$, $w_+ < \alpha$ is analogous. First notice that when $\mu = 0$ (1.1), (1.2), (1.3) possesses a unique solution

$$u_0(\xi) = \frac{(u_+ - u_-)\int_{-L}^{\xi} \exp(-\zeta^2/2\epsilon)d\zeta}{\int_{-L}^{L} \exp(-\xi^2/2\epsilon)d\xi} + u_-,$$

$$w_0(\xi) = \frac{(w_+ - w_-)\int_{-L}^{\xi} \exp(-\zeta^2/2\epsilon)d\zeta}{\int_{-L}^{L} \exp(-\xi^2/2\epsilon)d\xi} + w_-.$$

Also note that $w_0'(\xi) > 0$, $\xi \in [-L, L]$.

Now set $U(\xi) = u(\xi) - u_0(\xi)$, $W(\xi) = w(\xi) - w_0(\xi)$ and impose boundary conditions

$$U(-L) = U(L) = W(-L) = W(L) = 0. \tag{1.4}$$

If u, w are to solve (1.1), (1.2), (1.3), we see U, W must satisfy (1.4) and

$$\begin{aligned} \epsilon U'' &= \mu p(w_0 + W)' - \xi U', \\ \epsilon W'' &= \mu U' - \mu u_0' - \xi W'. \end{aligned} \tag{1.5}$$

Define the vectors

$$\mathbf{y}(\xi) = \begin{pmatrix} U(\xi) \\ W(\xi) \end{pmatrix}, \qquad \mathbf{f}(\xi, \mathbf{y}) = \begin{pmatrix} p(w_0 + W) \\ -U(\xi) - u_0 \end{pmatrix}.$$

Then (1.4), (1.5) has the form

$$\epsilon \mathbf{y}''(\xi) = \mu \mathbf{f}(\xi, \mathbf{y})' - \xi \mathbf{y}'(\xi), \tag{1.6}$$

$$\mathbf{y}(-L) = 0, \quad \mathbf{y}(L) = 0. \tag{1.7}$$

Let $\mathbf{v} \in C^1([-L, L]; \mathbb{R}^2)$. Define $\mathbf{T}$ to be the solution map that carries $\mathbf{v}$ into $\mathbf{y}$ where $\mathbf{y}$ solves

$$\epsilon \mathbf{y}''(\xi) = \mathbf{f}(\xi, \mathbf{v})' - \xi \mathbf{y}'(\xi), \tag{1.8}$$

$$\mathbf{y}(-L) = 0, \quad \mathbf{y}(L) = 0. \tag{1.9}$$

A straightforward computation shows that $\mathbf{y}(\xi)$ is given by the formula

$$\begin{aligned} \mathbf{y}(\xi) = & \mathbf{z} \int_{-L}^{\xi} \exp(-\zeta^2/2\epsilon) d\zeta + \frac{1}{\epsilon} \int_{-L}^{\xi} \mathbf{f}(\zeta, \mathbf{v}(\zeta) d\zeta \\ & - \frac{1}{\epsilon^2} \int_{-L}^{\xi} \int_0^{\zeta} \tau \mathbf{f}(\tau, \mathbf{v}(\tau)) \exp(\frac{\tau^2 - \zeta^2}{2\epsilon}) d\tau d\zeta \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} \mathbf{z} \int_{-L}^{L} \exp(-\zeta^2/2\epsilon) d\zeta = & -\frac{1}{\epsilon} \int_{-L}^{L} \mathbf{f}(\zeta, \mathbf{v}(\zeta)) d\zeta \\ & + \frac{1}{\epsilon^2} \int_{-L}^{L} \int_0^{\zeta} \tau \mathbf{f}(\tau, \mathbf{v}(\tau)) \exp(\frac{\tau^2 - \zeta^2}{2\epsilon}) d\tau d\zeta. \end{aligned} \tag{1.11}$$

Notice the fixed points of $\mu \mathbf{T}$ are solutions of (1.6), (1.7) which in turn yield solutions of (1.1), (1.2), (1.3).

It is clear that $\mathbf{T}$ maps $C^\circ([-L, L]; \mathbb{R}^2)$ continuously into $C^\circ([-L, L]; \mathbb{R}^2)$. Of course this implies $\mathbf{T}$ maps $C^1([-L, L]; \mathbb{R}^2)$ continuously into $C^\circ([-L, L]; \mathbb{R}^2)$. We now show $\mathbf{T}$ maps $C^1([-L, L]; \mathbb{R}^2)$ continuously into $C^1([-L, L]; \mathbb{R}^2)$.

For this purpose let $\mathbf{v}_1, \mathbf{v}_2 \in C^1([-L, L]; \mathbb{R}^2)$, $\mathbf{v}_1 = (U_1, W_1)$, $\mathbf{v}_2 = (U_2, W_2)$, and $\mathbf{y}_1 = \mu \mathbf{T} \mathbf{v}_1$, $\mathbf{y}_2 = \mu \mathbf{T} \mathbf{v}_2$. Differentiation of (1.10) shows

$$\begin{aligned} \mathbf{y}_1'(\xi) - \mathbf{y}_2'(\xi) = & (\mathbf{z}_1 - \mathbf{z}_2) \exp(-\xi^2/2\epsilon) \\ & + \frac{\mathbf{f}(\xi, \mathbf{v}_1(\xi))}{\epsilon} - \frac{\mathbf{f}(\xi, \mathbf{v}_2(\xi))}{\epsilon} \\ & - \frac{1}{\epsilon^2} \int_0^{\xi} \tau(\mathbf{f}(\tau, \mathbf{v}_1(\tau)) - \mathbf{f}(\tau, \mathbf{v}_2(\tau))) \exp(\frac{\tau^2 - \xi^2}{2\epsilon}) d\tau \end{aligned} \tag{1.12}$$

where $\mathbf{z}_1$, $\mathbf{z}_2$ are defined in the obvious manner.

Now let $\mathbf{v}_1, \mathbf{v}_2$ be in a finite ball B in $C^1([-L,L];\mathbb{R}^2)$. In particular for $\mathbf{v} = (U,W)$ in B, $w_0 + W$ is uniformly bounded in $\mathbb{R}$ and hence p is a uniformly continuous function of the argument $w_0 + W$. But for $\delta > 0$ arbitrary we know from uniform continuity of p that there is $l(\delta) > 0$ so that $|p(w_0(\xi) + W_1(\xi)) - p(w_0(\xi)+W_2(\xi))| < \delta$ if $|(W_1(\xi)+w_0(\xi)) - (W_2(\xi)+w_0(\xi))| < l(\delta)$, i.e. if $|W_1(\xi) - W_2(\xi)| < l(\delta)$. Hence $\sup_{-L<\xi<L} |p(w_0(\xi)+W_1(\xi)) - p(w_0(\xi)+W_2(\xi))| < \delta$ if $\sup_{-L<\xi<L} |W_1(\xi) - W_2(\xi)| < l(\delta)$ and so $\sup_{-L<\xi<L} |p(w_0(\xi) + W_1(\xi)) - p(w_0(\xi) + W_2(\xi))| \to 0$ as $\sup_{-L<\xi<L} |W_1(\xi) - W_2(\xi)| \to 0$. But this argument implies by the special nature of $\mathbf{f}(\xi, \mathbf{v}(\xi))$ that $\sup_{-L<\xi<L} \ |\mathbf{f}(\tau, \mathbf{v}_1(\tau)) - \mathbf{f}(\tau, \mathbf{v}_2(\tau))| \ \to 0$ as $\sup_{-L<\xi<L} \ |\mathbf{v}_1(\tau) - \mathbf{v}_2(\tau)| \ \to 0$. From (1.11), (1.12) we see then that $\sup_{-L<\xi<L} \ |\mathbf{y}_1'(\xi) - \mathbf{y}_2'(\xi)| \ \to 0$ as $\sup_{-L<\xi<L} \ |\mathbf{v}_1(\xi) - \mathbf{v}_2(\xi)| \ \to 0$ so $\mathbf{T}$ is a continuous map of $C^1([-L,L];\mathbb{R}^2)$ into itself.

Now note that (1.8) implies that if $\mathbf{v}$ is in a bounded set of $C^1([-L,L];\mathbb{R}^2)$, $\mathbf{y}$ will be in a bounded set of $C^2([-L,L];\mathbb{R}^2)$. This is because $\mathbf{f}(\xi,\mathbf{v}(\xi))'$ is uniformly bounded.

Hence $\mathbf{T}$ *is a continuous compact map of* $C^1([-L,L];\mathbb{R}^2)$ *into itself.*

Now define $\Omega = \{U, W \in C^1([-L,L];\mathbb{R}^2)$ such that $W(-L) + w_0(-L) < \alpha,\ \ W(L) + w_0(L) > \beta;\ W(\xi) + w_0'(\xi) > 0$ if $\alpha \le W(\xi) + w_0(\xi) \le \beta$; and $\sup_{L<\xi<L}[|U(\xi) + u_0(\xi)| + |W(\xi) + w_0(\xi)| + |U'(\xi) + u_0'(\xi)| + |W'(\xi) + w_0'(\xi)|\} < M + 1\}$. Ω is a *bounded set* in $C^1([-L,L];\mathbb{R}^2)$.

In addition Ω is *open*. To see this let $U, W \in \Omega$. Note the definition of Ω implies the set $A \overset{\text{def}}{=} \{\xi \in [-L,L];\ \alpha \le w_0(\xi) + W(\xi) \le \beta\}$ is a closed interval $[\xi_1,\xi_2]$. For if $A \ne [\xi_1,\xi_2]$ for some $\xi_1,\xi_2 \in [-L,L]$ means by the monotonicity of $w_0 + W$ on $[\xi_1,\xi_2]$ that for some $\bar{\xi} \notin [\xi_1,\xi_2]$ either $w_0(\bar{\xi}) + W(\bar{\xi}) = \alpha$ or $w_0(\bar{\xi}) + W(\bar{\xi}) = \beta$, with $w_0'(\bar{\xi}) + W'(\bar{\xi}) \le 0$ in either case. Of course this would imply $U, W \notin \Omega$, a contradiction.

So we have $A = [\xi_1,\xi_2]$ and denote $m = \min_{\xi\in A}(w_0'(\xi) + W'(\xi))$ which is positive. Since $w_0 + W \in C^1[-L,L]$ there is a larger interval $A_\delta \subseteq [-L,L]$, $A_\delta \supset A$, $A_\delta = [\xi_1 - \delta, \xi_2 + \delta]$ for some small $\delta > 0$, so that $w_0'(\xi) + W'(\xi) \ge \frac{m}{2}$ for $\xi \in A_\delta$.

Let $D = \min(\min_{\xi_2-\delta\le\xi\le L}(w_0(\xi) - W(\xi) - \beta), \min_{-L\le\xi\le\xi_1-\delta}(\alpha - w_0(\xi) - W(\xi))$. Since $w_0(\xi) - W(\xi) - \beta > 0$ on $[\xi_2 + \delta, L]$ and $\alpha - w_0(\xi) - W(\xi) > 0$ on $[-L, \xi_1 - \delta]$ we see $D > 0$.

Now let $\overline{U}, \overline{W}$ be such that

$$\sup_{-L\le\xi\le L}(|\overline{U}(\xi)| + |\overline{U}'(\xi)| + |\overline{W}(\xi)| + |\overline{W}'(\xi)|) < \nu,$$

where $\nu = \min(\frac{D}{2}, m/4)$. Consider $\xi \in [-L, L]$ for which $\alpha \leq w_0(\xi) + W(\xi) + \overline{W}(\xi) \leq \beta$. If we can show $w_0'(\xi) + W'(\xi) + \overline{W}'(\xi) > 0$ we will have proven Ω is open. But we see in this case that

$$w_0(\xi) + W(\xi) - \beta \leq -\overline{W}(\xi), \quad \alpha - w_0(\xi) - W(\xi) \leq \overline{W}(\xi),$$

and hence $w_0(\xi) + W(\xi) - \beta \leq D/2$, $\alpha - w_0(\xi) - W(\xi) \leq D/2$. But this implies by the definition of D that $\xi \in (\xi_1 - \delta, \xi_2 + \delta)$. So we have shown $\alpha \leq w_0(\xi) + W(\xi) + \overline{W}(\xi) \leq \beta$ implies $\xi \in A_\delta$. Now we compute at this value ξ:

$$w_0'(\xi) + W'(\xi) + \overline{W}'(\xi) \geq \frac{m}{2} + \overline{W}'(\xi) \geq m/4 > 0.$$

Hence Ω is open.

Now we recall a well known theorem of Leray-Schauder type (see for example Mawhin [10], Corollary IV.7).

Prop 1.2 Let X be a real normed vector space, Ω an open bounded subset of X, and $\mathbf{T}$ a compact map of X into itself. If zero is an interior point of Ω and $\phi \neq \mu \mathbf{T}\phi$ for all $\phi \in \partial\Omega$, $0 < \mu < 1$, then $\mathbf{T}$ has at least one fixed point in $\overline{\Omega}$.

In our problem we take $X = C^1([-L, L]; \mathbb{R}^2)$ and $\mathbf{T}$, Ω is as defined above. The origin in an interior point of Ω since the constraint $w_0'(\xi) + \overline{W}'(\xi) > 0$ is satisfies for all $\xi \in [-L, L]$ if $(\overline{U}, \overline{W})$ is a small $C^1([-L, L]; \mathbb{R}^2)$ perturbation. Note $\phi \in \partial\Omega$, $\phi = \mu \mathbf{T}\phi$, $\mu \in (0, 1)$, means that there is a solution $(u(\xi), w(\xi))$ of (1.1), (1.2), (1.3) which satisfies $w'(\xi) \geq 0$ if $\alpha \leq w(\xi) \leq \beta$ and either

(i) $w'(\xi_0) = 0, \quad \alpha \leq w(\xi_0) \leq \beta \quad$ for some $\xi_0 \in (-L, L)$

or

(ii) $\sup_{-L<\xi<L}\{|u(\xi)| + |w(\xi)| + |u'(\xi)| + |w'(\xi)|\} = M_0 + 1$ or both (i) and (ii).

Let us first consider possibility (i). In this case either $\alpha < w(\xi_0) < \beta$, $w(\xi_0) = \alpha$, or $w(\xi_0) = \beta$. We consider these cases separately.

Case 1: $\alpha < w(\xi_0) < \beta$, $w'(\xi_0) = 0$. In this case there are three possibilities, either $w''(\xi_0) < 0$, $w''(\xi_0) > 0$, or $w''(\xi_0) = 0$. If $w''(\xi_0) < 0$ then $w(\xi_0)$ is a local maximum which implies $w'(\xi) < 0$ for some $\xi < \xi_0$, $|\xi - \xi_0|$ small. But this implies $\alpha < w(\xi) < \beta$ and violates the requirement that $w'(\xi) \geq 0$. An analogous statement holds if $w''(\xi_0) > 0$ and now $w(\xi)$ is a local minimum. The case $w''(\xi_0) = 0$ is excluded since $w''(\xi_0) = 0$, $w'(\xi_0) = 0$ implies via (1.2) that $u'(\xi_0) = 0$. But in this uniqueness of solutions for (1.1), (1.2) as an initial value problem (see [7], Lemma 4.1) $u'(\xi_0) = 0$, $w'(\xi_0) = 0$ implies $u(\xi) = u(\xi_0)$, $w(\xi) = w(\xi_0)$ for all $\xi \in [-L, L]$ and hence we cannot satisfy (1.3), $w_- < \alpha$, $w_+ > \beta$.

Case 2: $w(\xi_0) = \alpha$, $w'(\xi_0) = 0$. In this case there are again the three canonical possibilities, $w''(\xi_0) < 0$, $w''(\xi_0) > 0$, or $w''(\xi_0) = 0$. We can immediately dismiss $w''(\xi_0) > 0$ and $w''(\xi_0) = 0$ for the same reasons as in Case 1. So we need only consider $w''(\xi_0) < 0$. In this case $w(\xi_0) = \alpha$ is a local maximum. Hence if we are to satisfy $w(L) = w_+ > \beta$ we must proceed through a local minimum at $\xi_1 > \xi_0$, i.e. $w(\xi_1) < \alpha$, $w'(\xi_1) = 0$, $w''(\xi_1) \geq 0$; $w(\xi) < \alpha$, $w'(\xi) < 0$, $\xi_0 < \xi \leq \xi_1$. Again $w''(\xi_1) = 0$ is impossible since that forces $u'(\xi_1) = 0$ and the uniqueness theorem ([7], Lemma 4.1) is contradicted. So we need only consider $w''(\xi_1) > 0$. From (1.2) we see $u'(\xi_1) > 0$, $u'(\xi_0) < 0$ which implies u has a local maximum at a point $\xi_0 < \zeta < \xi_1$, $u'(\zeta) = 0$, $u''(\zeta) \leq 0$, and again Lemma 4.1 of [7] tells us $u'(\zeta) < 0$. Since $p'(w) < 0$ for $w < \alpha$ this implies via (1.1) that $w'(\zeta) > 0$ which contradicts the fact that w is decreasing on (ξ_0, ξ_1). Hence $w''(\xi_0) < 0$ is excluded as well.

Case 3: $w(\xi_0) = \beta$, $w'(\xi_0) = 0$. Here again we see we can exclude $w''(\xi_0) < 0$ and $w''(\xi_0) = 0$ immediately. If $w''(\xi_0) > 0$ it follows that $w(\xi_0) = \beta$ is a local minimum so to satisfy $w(-L) = w_- < \alpha$ there must be $\xi_1 < \xi_0$ where $w(\xi_1) > \beta$ and w has a local maximum, $w(\xi) > \beta$ on (ξ_1, ξ_0). But the same reasoning as in Case 2 yields a contradiction.

From Cases 1,2,3 of (i) we see there is no solution of (1.1), (1.2), (1.3), $\mu \in (0,1)$, $(u(\xi) - u_0(\xi), w(\xi) - w_0(\xi))$ in Ω for which (i) can hold. So all solutions of (1.1), (1.2), (1.3), $\mu \in (0,1)$ in $\overline{\Omega}$ must satisfy $w'(\xi) > 0$ in $\alpha \leq w(\xi) \leq \beta$. But now the hypothesis of our theorem says (ii) cannot hold either. Thus we conclude from Prop 1.2 that (1.1), (1.2), (1.3) possesses a solution for which $u(\xi) - u_0(\xi)$, $w(\xi) - w_0(\xi)$ is in $\overline{\Omega}$.

To complete the proof we follow Dafermos [7] and extend the domains of u, w: Set

$$\begin{aligned} u(\xi; L) &= u_+, & w(\xi; L) &= w_+, & \xi &> L, \\ u(\xi; L) &= u_-, & w(\xi; L) &= w_-, & \xi &< -L. \end{aligned}$$

The extended pair $\{u(\cdot, L), w(\cdot, L)\}$ form a sequence in $C^\circ((-\infty, \infty); \mathbb{R}^2)$ and by virtue of the hypothesis of theorem we know $\sup_{-L<\xi<L}\{|u'(\xi; L)| + |w'(\xi; L)|\} \leq M$. So the sequence $\{(u(\cdot; L), w(\cdot; L)\}$ is precompact in $C^\circ((-\infty, \infty); \mathbb{R}^2)$. Thus there is a subsequence $L_n \to \infty$ as $n \to \infty$ since that $(u(\xi; L), w(\xi; L)) \to (u(\xi), w(\xi))$ uniformly as $n \to \infty$ on $(-\infty, \infty)$. As in Dafermos [7] $u(\xi), w(\xi)$ is a solution of P_ϵ and by its construction $w'(\xi) \geq 0$ if $\alpha \leq w(\xi) \leq \beta$. But by the same reasoning used in Cases 2,3, this connecting orbit must satisfy the more restrictive requirement $w'(\xi) > 0$ if $\alpha \leq w(\xi) \leq \beta$. This completes the proof of Theorem 1.1.

Having established Theorem 1.1 we can now proceed to weaken the hypothesis that the first derivatives of u, w be a priori bounded. This is done below.

Theorem 1.3 The conclusion of Theorem 1.1 remains valid if (H0) is replaced by the a priori estimate

$$\sup_{-L<\xi<L} (|u(\xi)| + |w(\xi)|) \leq M_1 \qquad (H1)$$

where again M_1 can depend on u_-, u_+, w_-, w_+, ϵ, p but is independent of μ and L.

Proof. All that we need to show is that if a solution of (1.1), (1.2), (1.3) satisfies **(H1)** it satisfies (H0). But this is precisely the nature of estimates (3.7), (3.8) given by Dafermos in [7]. For completeness we rederive these estimates.

Let $\mathbf{y}(\xi) = \binom{u(\xi)}{w(\xi)}$, $\mathbf{f}(\mathbf{y}) = \binom{p(w)}{-u}$. Then (1.1), (1.2), (1.3) have the form

$$\epsilon \mathbf{y}''(\xi) = \mu \mathbf{f}(\mathbf{y}(\xi))' - \xi \mathbf{y}'(\xi),$$

$$\mathbf{y}(-L) = \mathbf{y}_-, \quad \mathbf{y}(L) = \mathbf{y}_+$$

where $\mathbf{y}_- = (u_-, w_-)$, $\mathbf{y}_+ = (u_+, w_+)$.

We see easily that

$$\frac{d}{d\xi}(\exp(\xi^2/2\epsilon)\mathbf{y}'(\xi)) = \frac{\mu}{\epsilon}(\nabla \mathbf{f}(\mathbf{y})\mathbf{y}'(\xi)\exp(\frac{\xi^2}{2\epsilon}))$$

and hence by integrating from 0 to ξ we have

$$\exp(\epsilon^2/2\epsilon)\mathbf{y}'(\xi) - \mathbf{y}'(0) = \frac{\mu}{\epsilon}\int_0^\xi (\nabla \mathbf{f}(y)\mathbf{y}'(\zeta)\exp(\zeta^2/2\epsilon))d\zeta.$$

Since $\sup_{-L<\xi<L}(|u(\xi)| + |w(\xi)|) \leq M_1$ we know $\mu \ |\nabla \mathbf{f}(y)| \ \leq R$, R independent of L. So

$$|\exp(\xi^2/2\epsilon)\mathbf{y}'(\xi)| \ \leq \ |\mathbf{y}'(0)| + \frac{R}{\epsilon}\int_0^\xi |\mathbf{y}'(\zeta)| \ \exp(\frac{\zeta^2}{2\epsilon})d\zeta,$$

and using Gronwell's inequality we find

$$|\mathbf{y}'(\xi)| \ \leq \ |\mathbf{y}'(0)| \ \exp\left(\frac{2R|\xi| - \xi^2}{2\epsilon}\right).$$

But the function $(\frac{2R|\xi|-\xi^2}{2\epsilon})$ has a maximum value of $(\frac{R^2}{\epsilon})$ so we see

$$|\mathbf{y}'(\xi)| \ \leq | \ \mathbf{y}'(0)| \ \exp(\frac{R^2}{\epsilon}), \qquad -L \leq \xi \leq L,$$

where R is independent of L. So $\sup_{L<\xi<L} |\mathbf{y}'(\xi)|$ will be bounded independent of μ and L if $|\mathbf{y}'(0)|$ is bounded independent of μ and L.

Now we derive the bound on $|\mathbf{y}'(0)|$ First note

$$\mathbf{y}'(\xi) = \mathbf{z}\exp(-\xi^2/2\epsilon) + \frac{\mu}{\epsilon}\mathbf{f}(\mathbf{y}) - \frac{\mu}{\epsilon^2}\int_0^\xi \tau\mathbf{f}(\mathbf{y}(\tau))\exp\left(\frac{\tau^2-\xi^2}{2\epsilon}\right)d\tau$$

where

$$\mathbf{z}\int_{-L}^{L}\exp(-\zeta^2/2\epsilon) = \mathbf{y}(L) - \mathbf{y}(-L) - \frac{\mu}{\epsilon}\int_{-L}^{L}\mathbf{f}(\mathbf{y}(\tau))d\tau$$
$$+ \frac{\mu}{\epsilon^2}\int_{-L}^{L}\int_0^\zeta \tau\mathbf{f}(\mathbf{y}(\tau))\exp\left(\frac{\tau^2-\zeta^2}{2\epsilon}\right)d\tau\, d\zeta.$$

Now set $\xi = 0$, $L = 1$ in the above expressions. Then we have

$$\mathbf{y}'(0) = \mathbf{z} + \frac{\mu}{\epsilon}\mathbf{f}(\mathbf{y}(0)),$$
$$\mathbf{z}\int_{-1}^{1}\exp(-\zeta^2/2\epsilon) = \mathbf{y}(1) - \mathbf{y}(-1) - \frac{\mu}{\epsilon}\int_{-1}^{1}\mathbf{f}(\mathbf{y}(\tau))d\tau$$
$$+ \frac{\mu}{\epsilon^2}\int_{-1}^{1}\int_0^\zeta \tau\mathbf{f}(\mathbf{y}(\tau))\exp\left(\frac{\tau^2-\zeta^2}{2\epsilon}\right)d\tau\, d\zeta.$$

Since $\mu|\mathbf{f}(\mathbf{y}(\tau))| \leq$ constant (independent of μ and L) we see that $|\mathbf{y}'(0)|$ is bounded independent of μ and L. This completes the proof.

Theorem 1.3 gives a sufficient condition for solvability of P_ϵ when the boundary values w_- and w_+ are in different phases. We now give a result which applies to the case when w_-, w_+ are in the same phase.

Theorem 1.4 Assume $u_- > u_+$ and $w_-, w_+ < \alpha$ ($u_- < u_+$ and $w_-, w_+ > \beta$) and there is a constant M_2 such that every possible solution of (1.1), (1.2), (1.3) satisfies the a priori estimate

$$\sup_{-L<\xi<L}(|u(\xi)| + |w(\xi)|) \leq M_2. \qquad (H2)$$

Here M_2 can depend on u_-, w_+, ϵ, p but is independent of μ and L. Then their exist solutions of P_ϵ which satisfy the constraints $w(\xi) < \alpha$ ($w(\xi) > \beta$), i.e. *these special single phase data connecting orbit problems possesses single phase solutions.*

Proof. Theorem 1.4 is a special case of Thm 3.1 of [7].

2. The a priori estimates

In this section we derive the a priori estimates needed to apply Theorems 1.3 and 1.4. Before doing this we give some useful lemmas. The first is a result from [7] Thm. 4.1 or [8] Thm. 2.2.

Lemma 2.1 Let $(u(\xi), w(\xi))$ be a solution of (1.1), (1.2) on an interval $[-L, L]$, $\mu > 0$. Then on any subinteral (l_1, l_2) for which $p'(w(\xi)) < 0$ one of the following holds:

(i) $u(\xi)$ and $w(\xi)$ are constant on (l_1, l_2);

(ii) $u(\xi)$ is a strictly increasing (or decreasing) function with no critical points in $l_1, l_2)$; $w(\xi)$ has, at most, one critical point in (l_1, l_2) that necessarily must be a maximum (or minimum);

(iii) $w(\xi)$ is a strictly increasing (or decreasing) function with no critical point in (l_1, l_2); $u(\xi)$ has, at most, one critical point in (l_1, l_2), that necessarily must be a maximum (or minimum).

We will also need the following results.

Lemma 2.2 Let $(u(\xi), w(\xi))$ be a solution of (1.1), (1.2) on an interval $[-L, L]$, $\mu > 0$. Then on any subinterval (ℓ_1, ℓ_2) for which $p'(w(\xi)) > 0$ the graph of $u(\xi)$ versus $w(\xi)$ is convex at points where $w'(\xi) > 0$ and concave at points where $w'(\xi) < 0$.

Proof. We simply compute $\frac{d^2u}{dw^2}$ as follows:

$$\frac{du}{dw} = \frac{u'(\xi)}{w'(\xi)}; \qquad \frac{d}{d\xi}\left(\frac{u'(\xi)}{w'(\xi)}\right) = \left(\frac{u''(\xi)w'(\xi) - u'(\xi)w''(\xi)}{w'(\xi)^2}\right).$$

Now use (1.1), (1.2) to see that at $u(\xi), w(\xi)$ we have

$$\epsilon\frac{d^2u}{dw^2} = \frac{\mu(p'(w(\xi))w'(\xi)^2 + u'(\xi)^2)}{w'(\xi)^3} = \mu\left(p'(w) + (\frac{du}{dw})^2\right)\frac{1}{w'(\xi)}$$

which proves the result.

Lemma 2.3 Let $u(\xi), w(\xi)$ be a solution of (1.1), (1.2), (1.3) on an interval $[-L, L]$, $\mu > 0$ with $w'(\xi) > 0$ if $\alpha \leq w(\xi) \leq \beta$. Then u, w can have no local maxima or minima at points ξ for which $w(\xi) = \alpha$ or $w(\xi) = \beta$.

Proof. Since $w'(\xi) > 0$ if $\alpha \leq w(\xi) \leq \beta$ certainly w has no local maxima w minima at points where $w(\xi) = \alpha$. On the other hand if $u(\xi)$ has a local maximum

or minimum at such a point then $u'(\xi) = 0$ there and hence by (1.1) $u''(\xi) = 0$ as well. Differentiating (1.1) with respect to ξ we see $p''(\alpha) > 0$ and $p''(\beta) < 0$ implies $u'''(\xi) \neq 0$ at such points so u could not have taken on a local maximum or minimum.

We can use Lemmas 2.1, 2.2, 2.3 to prove the following useful statement regarding possible connecting orbits in the two phase data case. (Notice extrema of u are denoted by τ's, extrema of w are denoted by σ's.)

Lemma 2.4 Assume $w_- < \alpha$, $w_+ > \beta$ and let $u(\xi), w(\xi)$ be a possible solution of (1.1), (1.2), (1.3) with $\mu > 0$ for which $w'(\xi) > 0$ when $\alpha \leq w(\xi) \leq \beta$. Then one of the following holds:

(0) No extremal points: $u(\xi), w(\xi)$ have no local maxima or minima on $[-L, L]$. They are non-constant and monotone, w being monotone increasing.

(i) One extremal point: (a) $w(\xi)$ has a minimum at some σ_-, $w(\sigma_-) < w_-$; $u(\xi)$ is decreasing on $[-L, L]$.
(b) $w(\xi)$ has a maximum at some σ_+, $w(\sigma_+) > w_+$; $u(\xi)$ is increasing on $[-L, L]$.
(c) $u(\xi)$ has a maximum at some τ_- (or τ_+); $w(\tau_-) < \alpha$ (or $w(\tau_+) > \beta$) and $w(\xi)$ is increasing on $[-L, L]$.
(d) $u(\xi)$ has a minimum at some τ; $\alpha < w(\tau) < \beta$ and $w(\xi)$ is increasing on $[-L, L]$.

(ii) Two extremal points: (a) $u(\xi)$ has a local maximum at τ_- (or τ_+) and a local minimum at τ; $w(\xi)$ is increasing on $[-L, L]$ and $w_- < w(\tau_-) < \alpha$ or $w_+ > w(\tau_+) > \beta$, $\alpha < w(\tau) < \beta$.
(b) $w(\xi)$ has a minimum at σ_-, $w(\sigma_-) < w_-$; $u(\xi)$ has a local minimum at τ, $\tau > \sigma_-$, $\alpha < w(\tau) < \beta$.
(c)$w(\xi)$ has a maximum at σ_+, $w(\sigma_+) > w_+$; $u(\xi)$ has a local minimum at τ, $\tau < \sigma_+$, $\alpha < w(\tau) < \beta$.

(iii) Three extremal points: (a) $u(\xi)$ has local maxima at τ_-, τ_+, and a local minima at τ, $\tau_- < \tau < \tau_+$; $w(\zeta)$ is increasing with $w_- < w(\tau_-) < \alpha$, $\alpha < w(\tau) < \beta$, $\beta < w(\tau_+) < w_+$.
(b) $w(\xi)$ has a minimum at σ_-, $w(\sigma_-) < w_-$ and a maximum at σ_+, $w(\sigma_+) > w_+$ and $u(\xi)$ has a local minimal at τ, $\sigma_- < \tau < \sigma_+$, $\alpha < w(\tau) < \beta$.
(c) $w(\xi)$ has a minimum at σ_-, $w(\sigma_-) < w_-$, $u(\xi)$ has a local minimum at τ, $\alpha < w(\tau) < \beta$ and a local maximum at τ_+, $\beta < w(\tau_+) < w_+$, $\sigma_- < \tau < \tau_+$.
(d) $w(\xi)$ has a maximum at σ_+, $w(\sigma_+) > w_+$, $u(\xi)$ has a local maximum at τ_-, $w_- < w(\tau_-) < \alpha$, and a local minimum at τ, $\alpha < w(\tau) < \beta$.

Proof.

(0) <u>No extremal points:</u> The non-constancy follows from Lemma 4.1 of [7] and $w_- \neq w_+$.

(i) <u>One extremal point:</u> Either $u(\xi)$ or $w(\xi)$ is monotone. (a) If $u(\xi)$ is decreasing then $w(\xi)$ can have either a maximum or minimum. But Lemma 2.1 (ii) says it must be a minimum.
(b) If $u(\xi)$ is increasing the same reasoning as in (a) says $w(\xi)$ can possess only a maximum.
(c,d) On the other hand if $w(\xi)$ is monotone it must be monotone increasing since $w_- < w_+$. By Lemmas 2.1 (iii), 2.2, 2.3 we see the only possibilities are a maximum for u at τ_- (or τ_+) with $w(\tau_-) < \alpha$ or $w(\tau_+) > \beta$ or a minimum at τ with $\alpha < w(\tau) < \beta$.

(ii) <u>Two extremal points:</u> First consider the case of one local maxima and one local minima for $u(\xi)$. (a) Since $w(\xi)$ must be monotone increasing Lemmas 2.1 (iii) and 2.2 say the local maximum occurs where $w < \alpha$ or $w > \beta$ and the local minimum occurs where $\alpha < w < \beta$.

One local maxima and one local minima for $w(\xi)$ is impossible with $u(\xi)$ monotone. For if σ_1, σ_2 are such that w has a local maximum at σ_1 and a local minimum at σ_2 we must have $w'(\sigma_1) = 0$, $w''(\sigma_1) \leq 0$, $w'(\sigma_2) = 0$, $w''(\sigma_2) \geq 0$. Then (1.2) implies $u'(\sigma_1) \geq 0$, $u'(\sigma_2) \leq 0$.So monotonicity of u would yield u a constant. Lemma 4.1 of [7] would then give w a constant as well which contradicts $w_- \neq w_+$.
(b) If $w(\xi)$ has a minimum at σ_- and $u(\xi)$ has a local minimum at τ then certainly $w(\sigma_-) < w_-$. That means $\tau > \sigma_-$ and either $\alpha < w(\tau) < \beta$ or $w(\tau) > \beta$. But as $w'(\tau) > 0$ Lemma 2.1 (iii) says $w(\tau) > \beta$ is impossible.
(c) If $w(\xi)$ has a maximum at σ_+ and $u(\xi)$ has a local minimum at τ analagous reasoning to (b) above applies.

If $w(\xi)$ has a minimum at σ_-, $w(\sigma_-) < w_-$, u cannot have a maximum on $[-L, L]$. This is because such a maximum must occur at τ_1, $\tau_1 > \sigma_-$ implying $u'(\sigma_-) > 0$. This contradicts Lemma 2.1.(ii). Similarly, if $w(\xi)$ has a maximum at σ_+, $w(\sigma_+) > w_+$, u cannot have a maximum on $[-L, L]$. Again this is because such a maximum occurs at τ_1, $\tau_1 < \sigma_+$ implying $u'(\sigma_+) < 0$ contradicting Lemma 2.1.(ii).

(iii) <u>Three extremal points:</u> (a) First w cannot have three extreme points by Lemma 2.1.(ii) but u can. By Lemmas 2.1.(ii) and 2.2 we see they must go sequentially as a local maximum, local minimum, local maximum.
(b) If w has two extreme points one must be a minimum at σ_-, $w(\sigma_-) < w_-$ and the other a maximum at σ_+, $w(\sigma_+) > w_+$. Lemmas 2.1, 2.2 imply that the only possible extremal point for u is a minimum at τ, $\sigma_- < \tau < \tau_+$, $\alpha < w(\tau) < \beta$. If u has two extreme points and w has one then either
(c) w has a minimum at σ_-, $w(\sigma_-) < w_-$ or

(d) a maximum at σ_+, $w(\sigma_+) > w_+$. In (c) $w'(\xi) > 0$ for $\xi > \sigma_-$ so Lemma 2.1.(iii) says u must have a local maximum at τ_+, $\beta < w(\tau_+) < w_+$ and a local minimum at $\alpha < w(\tau) < \beta$, $\sigma_- < \tau < \sigma_+$. In (d) $w'(\xi) > 0$, $\xi < \sigma_+$ so again Lemma 2.1.(iii) says u must have a local maximum at τ_-, $w_- < w(\tau_-) < \alpha$ and a local minimum at τ, $\alpha < w(\tau) < \beta$, $\tau_- < \tau < \sigma_+$.

This completes all possible cases since extremal points at $w = \alpha$ or $w = \beta$ are excluded by Lemma 2.3.

Below are illustrated sketches in the $u - w$ plane of the possible cases described in Lemma 2.4.

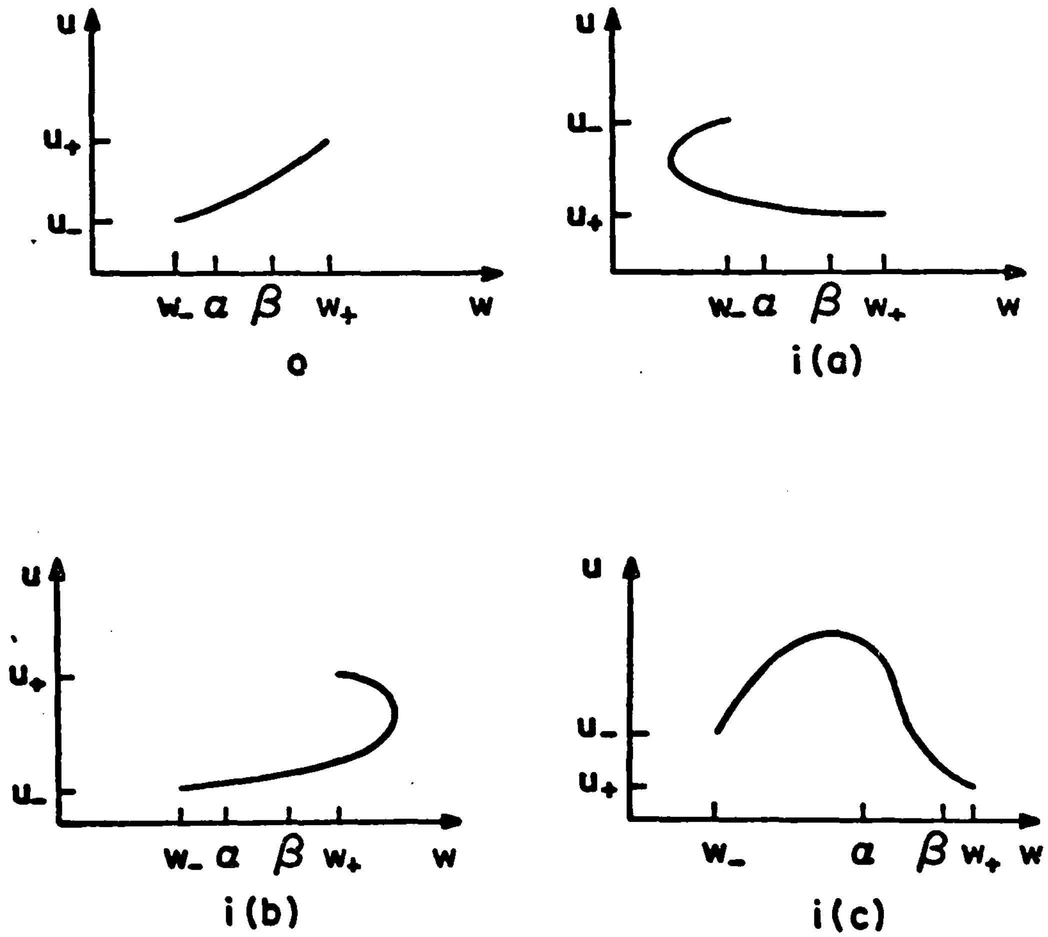


Figure 2.

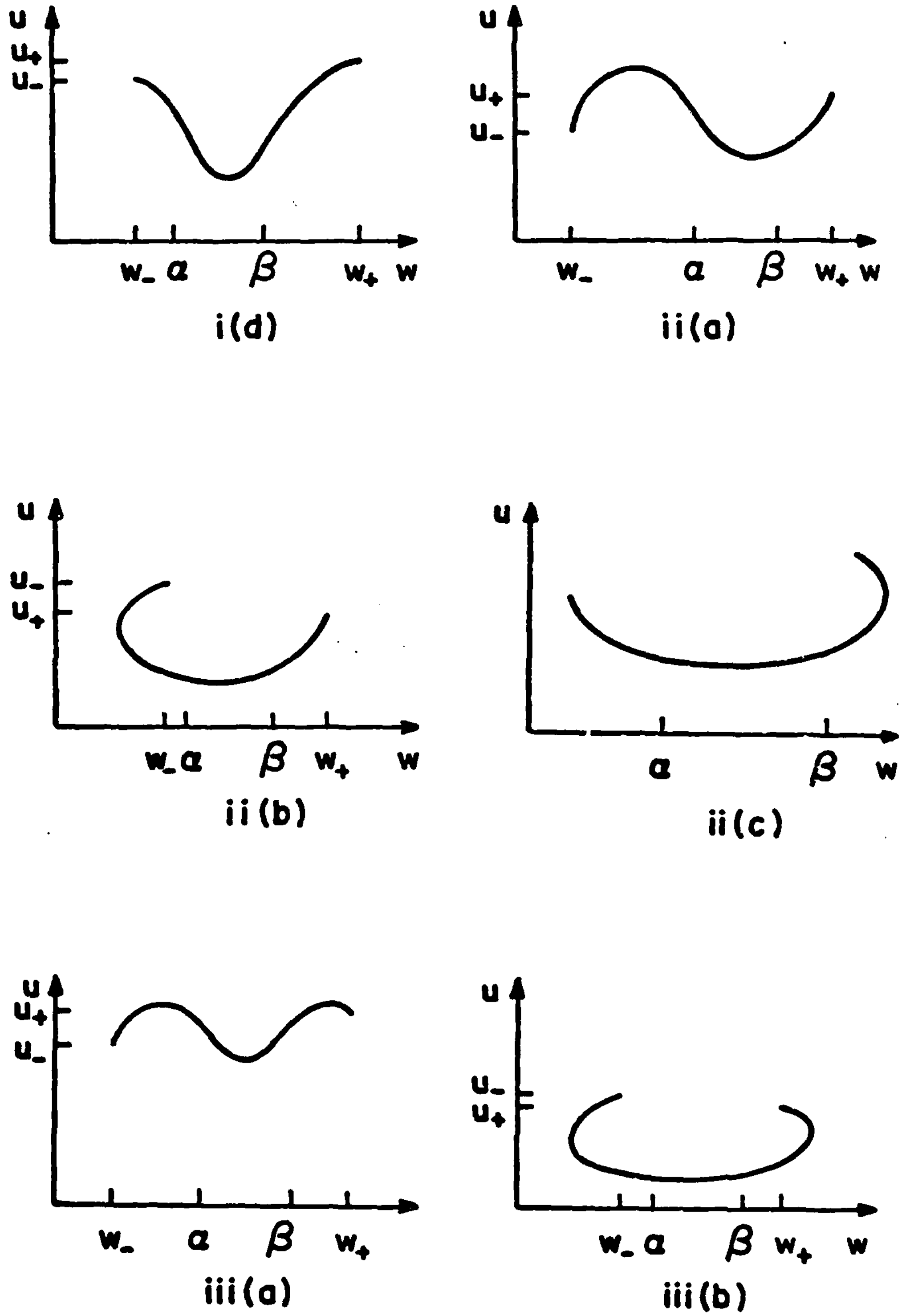
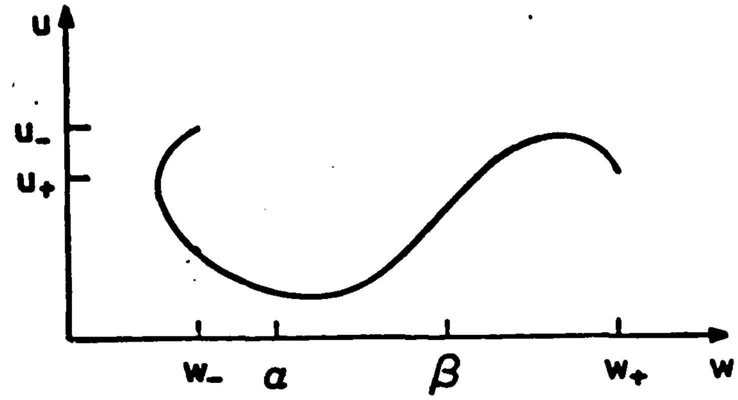


Figure 2 cont'd



iii (c)

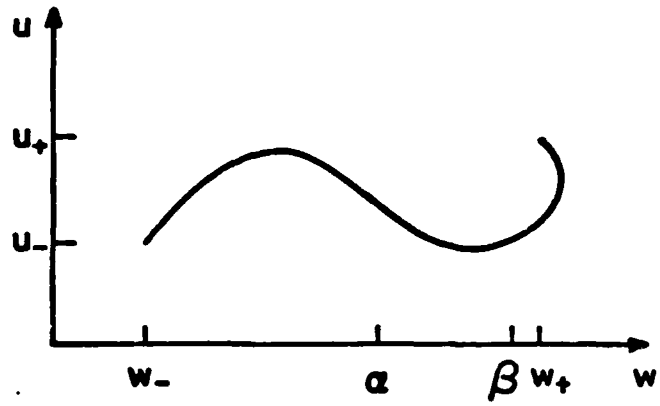


Figure 2 cont'd

iii (d)

Theorem 2.5 Assume $w_- < \alpha$, $w_+ > \beta$($w_- > \beta$, $w_+ < \alpha$). Then there is a constant M_1 such that every possible solution of (1.1), (1.2), (1.3), $0 \leq \mu \leq 1$, with $w'(\xi) > 0$ ($w'(\xi) < 0$) when $\alpha \leq w(\xi) \leq \beta$ satisfies the estimate

$$\sup_{-L<\xi<L} (|u(\xi)| + |w(\xi)|) \leq M_1. \tag{H1}$$

M_1 depends at most on u_-, u_+, w_-, w_+, ϵ, p and is independent of μ and L.

Theorem 2.6 Assume $u_+ < u_-$ and w_-, $w_+ < \alpha$ ($u_- < u_+$ and $w_-, w_+ > \beta$). Then there is a constant M_2 such that every possible solution of (1.1), (1.2), (1.3), $0 \leq \mu \leq 1$, satisfies the a priori estimate

$$\sup_{-L<\xi<L} (|u(\xi)| + |w(\xi)|) \leq M_2. \tag{H2}$$

M_2 depends at most on u_-, u_+, w_-, w_+, ϵ, p and is independent of μ and L.

Corollary 2.7 If $w_- < \alpha$, $w_+ > \beta$ ($w_- > \beta, w_+ < \alpha$) there exist solutions of P_ϵ which satisfy the constraints $w'(\xi) > 0$ ($w'(\xi) < 0$) when $\alpha \leq w(\xi) \leq \beta$, i.e.

the one phase change data connecting orbit problem possesses a one phase change solution.

If $u_+ < u$ and $w_-, w_+ < \alpha$ ($u_- < u_+$ and $w_-, w_+ > \beta$) there exist solutions of P_ϵ which satisfy the constraints $w(\xi) < \alpha$ ($w(\xi) > \beta$), i.e. *the single phase data connecting orbit problem possesses single phase solutions.*

Corollary 2.7, our main result, follows directly from Theorems 1.3, 1.4, 2.5, 2.6. So we pass on to verifying Theorems 2.5, 2.6.

Proof of Theorem 2.5 We consider the cases listed in Lemma 2.4 for the choice $w_- < \alpha$, $w_+ > \beta$. The proof for $w_- > \beta$, $w_+ < \alpha$ is similar and is omitted.

(0) No extremal points certainly implies the assertion of the theorem.

(ia) Since u is decreasing we have $u_+ \le u(\xi) \le u_-$. Since w has a minimum at σ_- we need only bound w from below. To do this we follow the method given in [7], Theorem 4.2.

Assume $\sigma_- \ge 0$ (similar arguments hold if $\sigma_- < 0$). Integrate (1.2) from σ_- to L and use $w'(\sigma_-) = 0$. Then

$$\epsilon w'(L) + \int_{\sigma_-}^{L} \xi w'(\xi) d\xi = -\mu u(L) + \mu u(\sigma_-).$$

Since $w'(L) > 0$ we have

$$\int_{\sigma_-}^{L} \xi w'(\xi) d\xi \le -\mu u_+ + \mu u(\sigma_-). \tag{2.1}$$

If $\zeta \ge \max\{1, \sigma_-\}$, then $w'(\xi) \le \xi w'(\xi)$ on (ζ, L) so that (2.1) implies

$$w(L) - w(\zeta) = \int_{\zeta}^{L} w'(\xi) d\xi \le \int_{\zeta}^{L} \xi w'(\xi) d\xi \le \int_{\sigma_-}^{L} \xi w'(\xi) d\xi \le -\mu u_+ + \mu u(\sigma_-)$$

and hence

$$w(\zeta) \ge w_+ + \mu u_+ - \mu u(\sigma_-). \tag{2.2}$$

Since $u_+ \le u(\sigma_-) \le u_-$, $0 \le \mu \le 1$, we see $w(\xi)$ is bounded below independently of μ and L if $1 \le \sigma_-$.

If $0 \le \sigma_- < 1$ integrate (1.2) from σ_- to ξ where $\sigma_- < \xi < 1$. Then we see

$$\epsilon w'(\xi) + \int_{\sigma_-}^{\xi} \zeta w'(\zeta) d\zeta = -\mu u(\xi) + \mu u(\sigma_-). \tag{2.3}$$

Since $w'(\xi) > 0$ on (σ_-, L) we see $\zeta w'(\zeta) > 0$ on (σ_-, ξ) and hence

$$\epsilon w'(\xi) \le -\mu u(\xi) + \mu u(\sigma_-), \qquad \sigma_- < \xi < 1. \tag{2.4}$$

Integrate (2.4) from σ_- to 1. We then see

$$\epsilon w(1) - \epsilon w(\sigma_-) \le \mu \int_{\sigma_-}^{1} (u(\sigma_-) - u(\xi)) d\xi$$

or

$$\epsilon w(1) - \mu \int_0^1 (u(\sigma_-) - u(\xi)) d\xi \le \epsilon w(\sigma_-). \tag{2.5}$$

Since $u_+ \le u(\xi) \le u_-$ and $w(1)$ is bounded from below by (2.2), (2.5) provides the desired bound from below for $w(\sigma_-)$ when $0 \le \sigma_- < 1$.

Cases i(b),i(c) are proven similarly and in fact i(a-c) fall into the cases treated in Theorem 4.2 of [7]. Case i(d) was not possible in [7] because of the assumption of hyperbolicity. Nevertheless the above method still works as we show below.

In case i(d) $w(\xi)$ is increasing so $w_- \le w(\xi) \le w_+$. Again assume $\tau \ge 0$ as the case $\tau < 0$ is similar. First integrate (1.1) τ to L. This yields

$$\epsilon u'(L) + \int_\tau^L \xi u'(\xi) d\xi = \mu p(w_+) - \mu p(w(\tau)). \tag{2.6}$$

Since $u'(L) > 0$ this implies

$$\int_\tau^L \xi u'(\xi) d\xi \le \mu p(w_+) - \mu p(w(\tau)). \tag{2.7}$$

If $\zeta \ge \max\{1, \tau\}$, since $u'(\xi) > 0$ on (ζ, L) we find $u'(\xi) \le \xi u'(\xi)$ on (τ, L) and

$$u_+ - u(\zeta) = \int_\zeta^L u'(\xi) d\xi \le \int_\zeta^L \xi u'(\xi) d\xi \le \int_\tau^L \xi u'(\xi) d\xi \le \mu p(w_+) - \mu p(w(\tau)),$$

$$u(\zeta) \ge u_+ - \mu p(w_+) + \mu p(w(\tau)). \tag{2.8}$$

Since $\alpha < w(\tau) < \beta$, we see for $1 \le \tau, u(\tau)$ is bounded from below independent of μ and L. Again if $0 \le \tau < 1$ integrate (1.1) from τ to ξ where $\tau < \xi < 1$. Then we see

$$\epsilon u'(\xi) + \int_\tau^\xi \zeta u'(\zeta) d\xi = \mu p(w(\xi)) - \mu p(w(\tau)).$$

Since $\zeta u'(\zeta) > 0$ on (τ, ξ) we find

$$\epsilon u'(\xi) \le \mu p(w(\xi)) - \mu p(w(\tau)),$$

and integrating from τ to 1 we see

$$\epsilon u(1) - \epsilon u(\tau) \leq \mu \int_\tau^1 p(w(\xi)) - p(w(\tau))d\xi$$

and

$$\epsilon u(1) - \mu \int_\tau^1 p(w(\xi)) - p(w(\tau))d\xi \leq \epsilon u(\tau). \tag{2.9}$$

We know $\max(u_-, u_+) \geq u(\xi)$ so u is bounded from above. Since $w(\xi)$ is bounded (2.8), (2.9) implies $u(\xi)$ is bounded from below on $[-L, L]$ independently of μ and L.

ii(a) Assume u has a local maximum at τ_-, $w(\tau_-) < \alpha$. (The case $w(\tau_+) > \beta$ is proved in a similar fashion.) Then the local minimum is at τ, $\tau_- < \tau$, $\alpha < w(\tau) < \beta$. For w we know $w_- \leq w(\xi) \leq w_+$. Trivially there are two possibilities we must consider:
<u>$\tau \geq 0$</u>. In this case proceed exactly as in the proof of i(d) above and we find <u>$u(\xi)$</u> bounded from below and certainly from above by u_+.
<u>$\tau < 0$</u>. If $\tau < 0$ then $\tau_- < 0$. We will show $u(\tau_-)$ is bounded from above. To do this consider the first case $\tau_- \leq -1$ and then the case $-1 \leq \tau_- \leq 0$ in a manner similar to i(d). This proves $u(\tau_-)$ will be bounded from above while it is certainly bounded from below by u_-.

So we find either $u(\tau)$ is bounded from below or $u(\tau_-)$ is bounded from above where the bounds are independent of μ and L. In the first case we use i(c) on $-L \leq \xi \leq \tau$ to bound $u(\tau_-)$ from above; in the second case we use i(d) on $\tau_- \leq \xi \leq L$ to bound $u(\tau)$ from below. Again the bounds will be independent of μ and L.

ii(b) If $\tau \geq 0$, we follow the argument of i(d) to note that $u(\tau)$ is bounded from below. Here we use the fact that $\alpha \leq w(\xi) \leq w_+$ for $\tau \leq \xi \leq L$. Since $u(\tau)$ is bounded from above by $\max(u_-, u_+)$ we know $u(\xi)$ is bounded from above and below. Finally we used result i(a) on $[-L, \tau]$ to see that w is also bounded from below at $\sigma_- \in (-L, \tau)$.

If $\tau < 0$ then using argument of i(d) again we find

$$u(\zeta) \geq u_- - \mu p(w_-) + \mu p(w(\tau)) \tag{2.10}$$

if $\zeta \leq \min\{-1, \tau\}$. But $\alpha < w(\tau) < \beta$ so (2.10) shows $u(\tau)$ is bounded from below if $\tau \leq -1$. If $-1 < \tau \leq 0$ then argument i(d) can be used again. We give the argument for completeness. First integrate (1.1) from τ to ξ where $\xi \in (-1, \tau)$. This yields

$$\epsilon u'(\xi) + \int_\tau^\xi \zeta u'(\zeta)d\zeta = \mu p(w(\xi)) - \mu p(w(\tau)). \tag{2.11}$$

On (ξ, τ), $\zeta u'(\zeta) > 0$ so the integral in (2.11) is negative. Thus we see

$$\epsilon u'(\xi) \geq \mu(p(w(\xi)) - p(w(\tau))). \tag{2.12}$$

Now integrate (2.12) from -1 to τ. This implies that

$$\epsilon u(\tau) \geq \epsilon u(-1) + \mu \int_{-1}^{\tau} p(w(\xi)) - p(w(\tau)) d\xi. \tag{2.13}$$

Now $w(\xi) \leq w(\tau)$ on $(-1, \tau)$ since $\alpha < w(\tau) < \beta$. Inspection of the graph of p (see Figure 3) shows that

$$p(w(\xi)) - p(w(\tau)) \geq p(\alpha) - p(\beta). \tag{2.14}$$

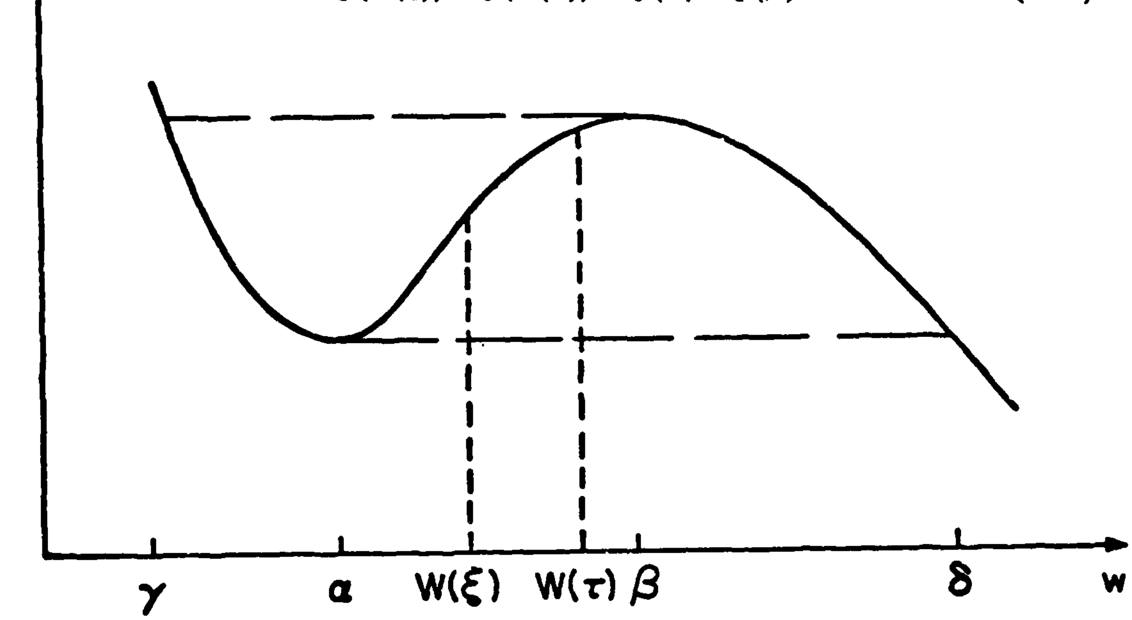


Figure 3.

(Notice $p(w(\xi)) - p(w(\tau))$ becomes positive if $w(\xi)$ decreases below γ.) Inserting (2.14) into (2.13) we find

$$\epsilon u(\tau) \geq \epsilon u(-1) + \mu(\tau + 1)(p(\alpha) - p(\beta))$$

and hence

$$\epsilon u(\tau) \geq \epsilon u(-1) + \mu(p(\alpha) - p(\beta)).$$

So $u(\tau)$ is bounded from if $\tau \leq 0$. Now use i(a) on $(-L, \tau)$ to see that $w(\sigma_-)$ is bounded from below.

ii(c) This case uses the same argument of ii(b). In this case we need

$$p(w(\tau)) - p(w(\xi)) \geq p(\alpha) - p(\beta) \tag{2.15}$$

when $w(\xi) \geq w(\tau)$. (It is *not* necessary that $p(w) \to \infty$ as $w \to \infty$.)

iii(a) In this case w is monotone increasing so $w_- \leq w(\xi) \leq w_+$ on $[-L, L]$. As to u either $\tau_+ \geq 0$ or it is not. If $\tau_+ \geq 0$ we use an argument of type in i(c) to conclude $u(\tau_+)$ is bounded from above. If $\tau_+ < 0$ then $\tau_- < 0$ and again we use an argument of the type in i(c) to see $u(\tau_-)$ is bounded from above. So if $\tau_+ \geq 0$, $u_+ \leq u(\tau_+) \leq$ bound from above; if $\tau_+ < 0$, then $u_- \leq u(\tau_-) \leq$ bound from above. But now we have reduced the problem to ii(a) which we have already considered.

iii(b) Either $\tau \geq 0$ or it is not. If $\tau \geq 0$ we use an argument as in ii(c) to find for $\tau \geq 1$

$$u(\tau) \geq u_+ - \mu p(w_+) + \mu(p(w(\tau)). \tag{2.16}$$

Since $\alpha \leq w(\tau) \leq \beta$ (2.16) shows $u(\tau)$ is bounded from below. If $0 \leq \tau < 1$ we again proceed as in ii(c) to see

$$\epsilon u(\tau) \geq \epsilon u(1) + \mu(p(\alpha) - p(\beta)).$$

So $u(\tau)$ is bounded from below.

If $\tau < 0$ we proceed as in (ii)(a) to again show $u(\tau)$ is bounded from below. So in either case $u(\tau)$ is bounded from above and below. We now use i(a) on $[-L, \tau]$ and i(b) on $[\tau, L]$ to show $w(\sigma_-)$ is bounded from below and $w(\sigma_+)$ is bounded from above. Of course in each case we use the fact that the value of w at the end point σ is bounded from above and below since $\alpha \leq w(\sigma) \leq \beta$.

iii(c) If $\tau \leq 0$ proceed as in ii(b). First note that if $\zeta \leq \min\{-1, \tau\}$ then

$$u(\zeta) \geq u_- + \mu(p(w_-) - p(w(\tau)).$$

Since $\alpha < w(\tau) < \beta$, $u(\tau)$ is bounded from below, if $\tau \leq -1$. If $-1 < \tau \leq 0$ we find

$$\epsilon u(\tau) \geq \epsilon u(-1) + \mu \int_{-1}^{\tau} p(w(\xi)) - p(w(\tau)) d\xi$$

where $w(\xi) \leq w(\tau)$, $-1 \leq \xi \leq \tau$. In this case figure 3 tells us

$$p(w(\xi)) - p(w(\tau)) \geq p(\alpha) - p(\beta)$$

so

$$\epsilon u(\tau) \geq \epsilon u(-1) + \mu(p(\alpha) - p(\beta))$$

and $u(\tau)$ is bounded from below for $\tau \leq 0$.

If $\tau \geq 0$ then $\tau_+ \geq 0$. We then estimate $u(\tau_+)$ from above as in i(c). So if $\zeta \geq \max\{\tau_+, 1\}$ we find

$$u(\zeta) \leq u_+ + \mu p(w(\tau_+)) - \mu p(w_+). \tag{2.17}$$

Since $\beta \leq w(\tau_+) \leq w_+$ we know $u(\tau_+)$ is bounded from above if $\tau_+ \geq 1$. If $0 \leq \tau_+ < 1$ we find

$$\epsilon u(\tau_+) \leq \epsilon u(1) - \mu \int_{\tau_+}^{1} p(w(\xi)) - p(w(\tau_+))d\xi. \tag{2.18}$$

But $\beta \leq w(\xi) \leq w_+$ for $\xi \in [\tau_+, 1]$ so the right hand side of (2.18) is bounded. So if $\tau \leq 0$ we see $u(\tau)$ is bounded from above and below; if $\tau > 0$ then $u(\tau_+)$ is bounded from above and below. Notice that for the second possibility we have reduced the problem to Case ii(b)which we have already considered. For the first possibility we apply i(a) on $[-L, \tau]$ to bound $w(\sigma_-)$ from below remembering the end point $w(\tau)$ lies in (α, β). Finally apply i(c) on $[\tau, L]$ to bound $u(\tau_+)$ from above again using the fact that $w(\tau) \in (\alpha, \beta)$.

iii(d) The proof is analagous to iii(c). This completes the proof of the theorem.

Proof of Theorem 2.6 In this case we never leave the hyperbolic regime $p'(w) < 0$. To see this consider the situation $w_- < w_+ < \alpha$. By Lemma 2.1 either $w(\xi)$ is monotone and hence we trivially have $w_- \leq w(\xi) \leq w_+$ or $u(\xi)$ is monotone decreasing and $w(\xi)$ possesses at most of critical point which must be a local minimum. Thus $w(\xi) < \alpha$ on $[-L, L]$. Now apply Thm 4.1 of [7]. The other cases are done analogously.

3. Existence of solutions to the Riemann problem: the case when $\{(u_\epsilon(\xi), w_\epsilon(\xi))\}$ are uniformly bounded

In this section we consider the applicability of the following result of Dafermos ([7], Thm. 3.2) to prove existence of solutions to the Riemann problem.

Prop. 3.1 For fixed $\epsilon > 0$, let $(u_\epsilon(\xi), w_\epsilon(\xi))$ denote a solution of P_ϵ. Suppose the set $\{(u_\epsilon(\xi), w_\epsilon(\xi)); 0 < \epsilon < 1\}$ is of uniformly bounded variation. Then $\{(u_\epsilon(\xi), w_\epsilon(\xi))\}$ possesses a subsequence which converges a.e. on $(-\infty, \infty)$ to function $(u(\xi), w(\xi))$ of bounded variation. The pair $u(\frac{x}{t})$, $w(\frac{x}{t})$ provide a weak solution of P.

In order to apply Prop. 3.1 we need the desired estimates on $\{(u_\epsilon(\xi), w_\epsilon(\xi))\}$ in both the two phase and one phase data case. First, however, we state an assumption on $p(w)$.

Assumption 3.2

(a) Assume $p(w) \to +\infty$ as $w \to -\infty$.
(b) Assume $p(w) \to -\infty$ as $w \to +\infty$.

Now we can state an existence theorem for the one phase data case.

Theorem 3.3 If $u_- > u_+$ and $w_-, w_+ < \alpha$ and Assumption 3.2(a) holds (or $u_+ > u_-$ and $w_-, w_+ > \beta$ and Assumption 3.2(b) holds) the sequence $\{(u_\epsilon(\xi), w_\epsilon(\xi)); 0 < \epsilon < 1\}$ as given by Corollary 2.7 possesses a subsequence which converges a.e. on $(-\infty, \infty)$ to function $(u(\xi), w(\xi))$ of bounded variation. The pair $u(\frac{x}{t})$, $w(\frac{x}{t})$ provides a solution to the Riemann problem with $w(\frac{x}{t}) < \alpha$ $(w(\frac{x}{t}) > \beta)$.

Proof. Since the sequence $\{(u_\epsilon(\xi), w_\epsilon(\xi))\}$ is such that $p'(w_\epsilon(\xi)) < 0$ *and either* $w_\epsilon(\xi) < \alpha$ *or* $w_\epsilon(\xi) > \beta$ Theorem 4.2(ii) of [7] combined with Assumption 3.2 (a) or (b) yields a uniform in ϵ, ξ bound for $\{u_\epsilon(\xi), w_\epsilon(\xi)\}$ on $(-\infty, \infty)$. Lemma 2.1 applied to the case $\mu = 1$, $-\infty < \xi < \infty$, shows $\{(u_\epsilon(\xi), w_\epsilon(\xi))\}$ is of uniformly bounded variation. Prop. 3.1 thus yields the result.

We now move on to the two phase data case.

From now on unless otherwise stated *we assume Assumption 3.2(a), (b) holds as well as the following condition of genuine nonlinearity.*

Assumption 3.3 $p''(w) > 0$ if $w \leq \alpha$; $p''(w) < 0$ if $w \geq \beta$.

We now proceed to give a sequence of lemmas based on Lemma 4.1 [8] that will help us in our search for an estimate on $\{u_\epsilon(\xi) w_\epsilon(\xi)\}$. Here and *for the rest of this paper* $\{u_\epsilon(\xi), w_\epsilon(\xi); 0 < \epsilon < 1\}$ *denotes the solution of* P_ϵ *given by Corollary 2.7 when* $w_- < \alpha, w_+ > \alpha$. (Results for the case $w_- > \beta$, $w_+ < \beta$ can always be obtained by analagous arguments given for the $w_- < \alpha$, $w_+ > \beta$ case.)

Lemma 3.4 The list of possible graphs for $(u_\epsilon(\xi), w_\epsilon(\xi))$ given in Lemma 2.4 is still valid when $L = \infty$.

Proof. The argument given in the finite L case still applies.

Notation. Points of minima, maxima of $u_\epsilon(\xi), w_\epsilon(\xi)$ are denoted as in Lemma 2.4 with the addition of a superscript ϵ to emphasize the dependence on ϵ.

Lemma 3.5 In cases 0, $i(a, b, c)$ of Lemma 2.4 $(u_\epsilon(\xi), w_\epsilon(\xi))$ are uniformly bounded independent of ϵ on $(-\infty, \infty)$, i.e. there exists a constant N depending

at most on u_-, u_+, w_-, w_+, p and independent of ϵ, $0 < \epsilon < 1$, such that

$$\sup_{-\infty<\xi<\infty} |u_\epsilon(\xi)| + |w_\epsilon(\xi)| \le N. \tag{3.1}$$

Proof.

0. Obvious.

i(a). In this case $u_\epsilon(\xi)$ is monotone decreasing and hence $u_+ \le u(\xi) \le u_-$ on $(-\infty,\infty)$. Denote $\frac{dw_\epsilon(\xi)}{du} = \frac{w_\epsilon'(\xi)}{u_\epsilon'(\xi)}$. We claim $0 < \frac{dw_\epsilon(\xi)}{du} < \left(\frac{-1}{p'(w_\epsilon(\xi))}\right)^{1/2}$ on $(-\infty, \sigma_-^\epsilon]$. For if not set

$$\xi_1 = \max\{\xi \in (-\infty, \sigma_-^\epsilon]; \quad \frac{dw_\epsilon(\xi)}{du} \ge +\left(\frac{-1}{p'(w_\epsilon(\xi))}\right)^{1/2}\}.$$

Since w_ϵ has its minimum at σ_-^ϵ, $\frac{dw_\epsilon}{du} = 0$ there and so $\xi_1 < \sigma_-^\epsilon$ must exist. A direct computation shows

$$\epsilon \frac{d}{d\xi}\left(\frac{dw_\epsilon}{du}(\xi)\right) = -1 - p'(w_\epsilon(\xi))\left(\frac{dw_\epsilon}{du}\right)^2 \tag{3.2}$$

so $\frac{d}{d\xi}\left(\frac{dw_\epsilon}{du}(\xi)\right) = 0$ at $\xi = \xi_1$. Furthermore, by the definition ξ_1 we have $0 < \frac{dw_\epsilon}{du} < +\left(-\frac{1}{p'(w_{\epsilon(\xi)})}\right)^{1/2}$ on $(\xi_1, \sigma_-^\epsilon)$ and thus $\frac{d}{d\xi}\left(\frac{dw_\epsilon}{du}(\xi)\right) < 0$ on $(\xi_1, \sigma_-^\epsilon)$ and $\frac{d^2}{d\xi^2}\left(\frac{dw_\epsilon}{du}(\xi)\right) < 0$ at $\xi = \xi_1$. On the other hand differentiation of (3.2) shows

$$\epsilon \frac{d^2}{d\xi^2}\left(\frac{dw_\epsilon}{du}(\xi)\right) = -p''(w_\epsilon(\xi))w_\epsilon'(\xi)\left(\frac{dw_\epsilon}{du}(\xi)\right)^2 \qquad \text{at } \xi = \xi_1.$$

Since Assumption 3.3 implies $p''(w_\epsilon(\xi_1)) > 0$ and we know $w_\epsilon'(\xi_1) < 0$ we have $\epsilon\frac{d^2}{d\xi^2}\left(\frac{dw_\epsilon}{du}(\xi)\right) > 0$ at $\xi = \xi_1$, a contradiction. So we see $\frac{d}{d\xi}\left(\frac{dw_\epsilon}{du}(\xi)\right) \le 0$ on $(-\infty, \sigma_-^\epsilon]$. Hence for any $\xi \in (-\infty, \sigma_-^\epsilon]$, $\frac{dw_\epsilon}{du}(\xi) < \frac{dw_\epsilon}{du}(-\infty) = +\left(-\frac{1}{p'(w_-)}\right)^{1/2}$. Now let us compute

$$w_\epsilon(\sigma_-^\epsilon) - w_- = \int_{u_-}^{u_\epsilon(\sigma_-^\epsilon)} \frac{dw_\epsilon}{du} du =$$

$$\int_{u_\epsilon(\sigma_-^\epsilon)}^{u_-} \left(-\frac{dw_\epsilon}{du}\right) du \ge -\int_{u_\epsilon(\sigma_-^\epsilon)}^{u_-} \left(-\frac{1}{p'(w_-)}\right)^{1/2} du$$

and thus

$$w_\epsilon(\sigma_-^\epsilon) - w_- \ge -\left(-\frac{1}{p'(w_-)}\right)^{1/2}(u_- - u_\epsilon(\sigma_-^\epsilon)). \tag{3.3}$$

Since $u_+ \le u_\epsilon(\sigma_-^\epsilon) \le u_-$ (3.3) shows $w_\epsilon(\sigma_-^\epsilon)$ is bounded from below independent of ϵ.

i(b) The proof is similar to i(a).

i(c) In this case $w_\epsilon(\xi)$ is monotone increasing so $w_- \le w_\epsilon(\xi) \le w_+$ on $(-\infty, \infty)$. Denote $\frac{du_\epsilon}{dw}(\xi) = \frac{u_\epsilon'(\xi)}{w_\epsilon'(\xi)}$. We claim $0 < \frac{du_\epsilon}{dw} < +(-p'(w_\epsilon(\xi))^{1/2}$ on $(-\infty, \tau_-^\epsilon]$. For if not set

$$\xi_1 = \max\{\xi \in (-\infty, \tau_-^\epsilon]; \quad \frac{du_\epsilon}{dw}(\xi) \ge +(-p'(w_\epsilon(\xi))^{1/2}\}$$

where $u_\epsilon(\xi)$ has its maximum at τ_-^ϵ, $w_\epsilon(\tau_-^\epsilon) < \alpha$. (The case when the maximum is at τ_+^ϵ, $w_\epsilon(\tau_+^\epsilon) > \beta$ is done similarly). Since $u_\epsilon(\xi)$ has a local maximum at τ_-^ϵ where $\frac{du_\epsilon}{dw}(\xi) = 0$, ξ_1 must exist with $\xi_1 < \tau_-^\epsilon$.

A direct computation shows

$$\epsilon \frac{d}{d\xi}\left(\frac{du_\epsilon}{dw}(\xi)\right) = p'(w_\epsilon(\xi)) + \left(\frac{du_\epsilon}{dw}(\xi)\right)^2 \tag{3.4}$$

so $\frac{d}{d\xi}\left(\frac{du_\epsilon}{dw}(\xi)\right) = 0$ at $\xi = \xi_1$. Furthermore by the definition of ξ_1 we have $0 < \frac{du_\epsilon}{dw}(\xi) < +(-p'(w_\epsilon(\xi)))^{1/2}$ on (ξ_1, τ_-^ϵ) and thus $\frac{d}{d\xi}\left(\frac{du_\epsilon}{dw}(\xi)\right) < 0$ on (ξ_1, τ_-^ϵ). So we have $\frac{d^2}{d\xi^2}\left(\frac{du_\epsilon}{dw}(\xi)\right) < 0$ at $\xi = \xi_1$. On the other hand differentiation of (3.4) shows $\epsilon \frac{d^2}{d\xi^2}\left(\frac{du_\epsilon}{dw}(\xi)\right) = p''(w_\epsilon(\xi))w_\epsilon'(\xi) > 0$ at ξ_1 and we have $\epsilon \frac{d^2}{d\xi^2}\left(\frac{du_\epsilon}{dw}(\xi)\right) > 0$ at $\xi = \xi_1$, a contradiction. So we see $\frac{d}{d\xi}\left(\frac{du_\epsilon}{d\xi}(\xi)\right) < 0$ on $(-\infty, \tau_-^\epsilon]$ and hence for any $\xi \in (-\infty, \tau_-^\epsilon]$, $0 < \frac{du_\epsilon}{dw}(\xi) < \frac{du_\epsilon}{dw}(-\infty) = +(-p'(w_-))^{1/2}$. Now we compute

$$u_\epsilon(\tau_-^\epsilon) - u_- = \int_{w_-}^{w(\tau_-^\epsilon)} \frac{du_\epsilon}{dw} dw \le (-p'(w_-))^{1/2}(w(\tau_-^\epsilon) - w_-).$$

Since $w_- \le w(\tau_-^\epsilon) \le w_+$ we see $u_\epsilon(\tau_-^\epsilon)$ is bounded from above independent of ϵ for $w(\tau_-^\epsilon) < \alpha$. As noted above analagous reasoning shows that if $w(\tau_+^\epsilon) > \beta$ we have

$$u_\epsilon(\tau_+^\epsilon) \le u_+ + (-p'(w_+))^{1/2}(w_+ - w_\epsilon(\tau_+^\epsilon))$$

and since $w_- \le w_\epsilon(\tau_+^\epsilon) \le w_+$ an ϵ independent bound on $u_\epsilon(\tau_+^\epsilon)$ is produced.

Lemma 3.6 Let τ^ϵ denote the points where $u_\epsilon(\xi)$ takes on its local minimum, $\alpha < w_\epsilon(\tau^\epsilon) < \beta$. If there is a subsequence $\{\tau^{\epsilon_n}\}$ of $\{\tau^\epsilon\}$, $\epsilon_n \to 0+$ such that either

(a) $\tau^{\epsilon_n} \geq m > 0$ or $\tau^{\epsilon_n} \leq -m > 0$, m a constant independent of ϵ, or

(b) $u_\epsilon(\tau^{\epsilon_n})$ is bounded from below independent of ϵ,

then for Case i(c) $\{(u_{\epsilon_n}(\xi), w_{\epsilon_n}(\xi))\}$ satisfies (3.1).

Proof. Assume $\tau^{\epsilon_n} \leq -m > 0$. Then $u'_{\epsilon_n}(\xi) \leq 0$ on $(-\infty, \tau^{\epsilon_n}]$ and $\xi u'_{\epsilon_n}(\xi) \geq -mu'_{\epsilon_n}(\xi)$ on $(-\infty, \tau^{\epsilon_n}]$. Integrate (0.5) from $-\infty$ to τ^{ϵ_n} and we see

$$-m(u_\epsilon(\tau^{\epsilon_n}) - u_-) \leq \int_{-\infty}^{\tau^{\epsilon_n}} \xi u'_{\epsilon_n}(\xi) d\xi = p(w(\tau^{\epsilon_n})) - p(w_-)$$

and hence

$$\frac{1}{m}(-p(w_\epsilon(\tau^{\epsilon_n}) + p(w_-)) - u_- \leq u(\tau^{\epsilon_n}). \tag{3.5}$$

Since $w_\epsilon(\xi)$ is monotone $w_- \leq w_\epsilon(\xi) \leq w_+$ and we see $u_\epsilon(\tau^{\epsilon_n})$ is bounded from below independent of ϵ. The case $\tau^{\epsilon_n} \geq m > 0$ is done similarly. So in (a) or (b) $u(\tau^{\epsilon_n})$ is bounded from below and hence $\{(u_{\epsilon_n}(\xi), w_{\epsilon_n}(\xi)); 0 < \epsilon_n < 1\}$ satisfies (3.1).

Lemma 3.7 In Cases ii(a), (b), (c), iii(a), (b), (c), (d) assume $\{\tau^\epsilon\}$ satisfies the hypothesis of Lemma 3.6. Then $\{(u_{\epsilon_n}(\xi), w_{\epsilon_n}(\xi)); 0 < \epsilon_n < 1\}$ satisfies (3.1).

Proof ii(a). Let $\tau_-^{\epsilon_n}$ denote the point where $u_{\epsilon_n}(\xi)$ has its local maximum, $w_{\epsilon_n}(\tau_-^{\epsilon_n}) < \alpha$. The method of proof for i(c) in Lemma 3.5 shows $u_{\epsilon_n}(\tau_-^{\epsilon_n})$ is bounded from above independent of ϵ_n; $u_\epsilon(\tau_-^\epsilon)$ is bounded from below by u_-.

If $\tau^{\epsilon_n} < -m < 0$ we know $\xi u'_{\epsilon_n}(\xi) \geq -mu'_{\epsilon_n}(\xi)$ on $(\tau_-^{\epsilon_n}, \tau^{\epsilon_n})$. Integrate (0.5) from $\tau_-^{\epsilon_n}$ to τ^{ϵ_n}. We see

$$-m(u_{\epsilon_n}(\tau^{\epsilon_n}) - u_{\epsilon_n}(\tau_-^{\epsilon_n})) \leq p(w_{\epsilon_n}(\tau^{\epsilon_n})) - p(w_{\epsilon_n}(\tau_-^{\epsilon_n})). \tag{3.6}$$

Inequality (3.6)combined with the monotonicity of $w_{\epsilon_n}(\xi)$ gives the bound on $u_{\epsilon_n}(\tau^{\epsilon_n})$ from below. If $\tau^{\epsilon_n} > m > 0$ integration of (0.5) from τ^{ϵ_n} to ∞ produces the bound from below on $u_{\epsilon_n}(\tau^{\epsilon_n})$. So if the hypothesis of Lemma 3.6 holds, $\{(u_{\epsilon_n}(\xi), w_{\epsilon_n}(\xi)), 0 < \epsilon_n < 1\}$ satisfies (3.1).

ii(b). First consider the case when $\tau^{\epsilon_n} < -m < 0$. Proceed as in the proof of Lemma 3.6 to (3.5). Since $\alpha < w_{\epsilon_n}(\tau^{\epsilon_n}) < \beta$ (3.5) delivers a bound on $u_{\epsilon_n}(\tau^{\epsilon_n})$ from below. Now use the method of proof of Lemma 3.5 1(a) to bound $w_{\epsilon_n}(\xi)$ from below. If $\tau^{\epsilon_n} > m > 0$ (or $u_{\epsilon_n}(\tau^{\epsilon_n})$ is already bounded from below) an analagous argument works.

ii(c). Proceed as in the proof ii(b) above only now use the argument of Lemma i(b) to bound $w_{\epsilon_n}(\xi)$ from above.

iii(a). First bound the two local maxima of $u_{\epsilon_n}(\xi)$ from above as in the proof of Lemma 3.5 i(c). Now follow the proof of ii(a) above to bound $u_{\epsilon_n}(\tau^{\epsilon_n})$ from below.

iii(b). Proceed as in the proof of Lemma 3.6 to bound $u_{\epsilon_n}(\tau^{\epsilon_n})$ from below. Then bound $w_{\epsilon_n}(\sigma_-^{\epsilon_n})$ from below and $w_{\epsilon_n}(\sigma_+^{\epsilon})$ from above via the proof of Lemma 3.5 i(a),(b).

iii(c). First bound the local maximum of $u_{\epsilon_n}(\xi)$ from above by the method of Lemma 3.5 i(c). Then bound the local minimum $u_{\epsilon_n}(\tau^{\epsilon_n})$ from below by the method of ii(a) above. Now use the method of Lemma 3.5 i(a) to bound $w_{\epsilon_n}(\sigma_-^{\epsilon_n})$ from below.

iii(d). The proof is analagous to iii(c).

We are now in a position to state the main results of this section.

Theorem 3.8 Assume $w_- < \alpha$, $w_+ > \beta$ (or $w_- > \alpha$, $w_+ < \beta$) and let $(u_\epsilon(\xi), w_\epsilon(\xi))$ denote the solution of P_ϵ given by Corollary 2.7. Let Assumptions 3.2, 3.3 and the hypothesis of Lemma 3.6 hold. Then $\{u_{\epsilon_n}(\xi), w_{\epsilon_n}(\xi); 0 < \epsilon_n < 1\}$ possesses a subsequence which converges a.e. on $(-\infty, \infty)$ to a function $u(\xi), w(\xi)$ *of bounded variation. The pair* $u(\frac{x}{t})$, $w(\frac{x}{t})$ *provides a solution to the Riemann problem.*

Proof. If $w_- < \alpha$, $w_+ > \beta$ use Lemmas 3.5, 3.6, 3.7 and Prop. 3.1. If $w_- > \beta$, $w_+ < \alpha$ we can prove a similar set of lemmas to Lemma 3.5, 3.6, 3.7 and again use Prop. 3.1.

Remark 3.9 If the hypothesis of Lemma 3.6 does *not* hold then $\tau^\epsilon \to 0$, $u_\epsilon(\tau^\epsilon) \to -\infty$ as $\epsilon \to 0+$.

Proof. If $\tau^\epsilon \not\to 0$ as $\epsilon \to 0+$ then there is a subsequence $\{\tau^{\epsilon_k}\}$ so that $|\tau^{\epsilon_k}| > m > 0$, m a positive constant independent of ϵ_k. From this subsequence we can extract another subsequence so that either $\tau^{\epsilon_n} \geq m > 0$ or $\tau^{\epsilon_n} \leq -m < 0$, a contradiction. On the other hand if $\tau^\epsilon \to 0$ and $u_\epsilon(\tau^\epsilon) \not\to -\infty$ then of course the hypothesis of Lemma 3.6 holds.

From Remark 3.9 we see that the only situation which may cause difficulty vis-a-vis solvability of the Riemann problem (at least under Assumptions 3.2, 3.3) is when $\tau^\epsilon \to 0$, $u_\epsilon(\tau^\epsilon) \to -\infty$ as $\epsilon \to 0+$. This possibility is the subject of the next section. Of course if we were to make the hypotheses that $u_\epsilon(\xi)$, is uniformly bounded independent of ϵ, $0 < \epsilon < 1$, when $\alpha \leq w_\epsilon(\xi) \leq \beta$ then existence of a solution to the Riemann problem follows trivially from Lemma 3.6.

4. Existence of solutions to the Riemann problem: the case when $u_\epsilon(\tau^\epsilon) \to -\infty$ as $\tau^\epsilon \to 0$

In this section we discuss the possible consequences of the case when $u_\epsilon(\tau^\epsilon) \to -\infty$ as $\tau^\epsilon \to 0$. (We use the notation of Section 3.)

Our first goal is to show $u_\epsilon(\xi), w_\epsilon(\xi)$ has a pointwise a.e. limit. To do this we need a sequence of lemmas. The first one is modeled on Thm. 2.3 of [S]. *We let Assumptions 3.2, 3.3 hold in this section.*

Lemma 4.1 Let $(u_\epsilon(\xi), w_\epsilon(\xi))$ be a solution of P_ϵ as given by Corollary 2.7 when $w_- < \alpha$, $w_+ > \beta$. Let $\overline{u} = \min(u_-, u_+)$. Then if $u_\epsilon(\xi)$ has a local minimum at τ^ϵ with $\alpha < w_\epsilon(\tau^\epsilon) < \beta$ (as in Cases i(d), ii(a),(b),(c), iii(a),(b),(c),(d) of Lemma 2.4 with $L = \infty, \mu = 1$) we have the estimates

$$N_0(\sigma_2 - \sigma_1) \geq \int_{\sigma_1}^{\sigma_2} u_\epsilon(\xi)d\xi \geq \overline{u}(\sigma_2 - \sigma_1) - (p(\beta) - p(\alpha), \tag{4.1}$$

$$\overline{u} - \frac{(p(\beta) - p(\alpha))}{|\xi - \tau^\epsilon|} \leq u_\epsilon(\xi) \leq N_0, \qquad -\infty < \xi < \infty. \tag{4.2}$$

Here $(\sigma_1, \sigma_2) \subset (-\infty, \infty)$ and N_0 is a constant independent of ϵ.

Proof. The bound from above on $u_\epsilon(\xi)$ in (4.1), (4.2) follows from the proof of Lemmas 3.5, 3.6, 3.7. So we now proceed to get the bounds from below. We first check i(d). Fix $\ell < \infty$ sufficiently large so that $w_\epsilon(-\ell) < \alpha$, $w_\epsilon(\ell) > \beta$. Assume for the moment $u_\epsilon(-\ell) \leq u_\epsilon(\ell)$, and let $\theta > -\ell$ be such that $u_\epsilon(\theta) = u_\epsilon(-\ell)$. Then as shown in Figure 4 we have $u_\epsilon(\xi) \leq u_\epsilon(-\ell)$ on $(-\ell, \theta)$, $u_\epsilon(\xi) \geq u_\epsilon(-\ell)$ on $\theta < \xi < \ell$ when $-\ell < \tau^\epsilon < \theta < \ell$. From (0.10) we know

$$\epsilon(u_\epsilon(\xi) - u_\epsilon(-\ell))'' + \xi(u_\epsilon(\xi) - u_\epsilon(-\ell))' = p(w_\epsilon)' \tag{4.3}$$

and integration of (4.3) from $-\ell$ to θ shows

$$\epsilon u_\epsilon'(\theta) - \epsilon u_\epsilon'(-\ell) - \int_{-\ell}^{\theta} (u_\epsilon(\xi) - u_\epsilon(-\ell))d\xi = p(w_\epsilon(\theta)) - p(w_\epsilon(-\ell)). \tag{4.4}$$

But $u'(\theta) > 0$, $u'(-\ell) < 0$ and hence we have

$$\int_{-\ell}^{\theta} (u_\epsilon(-\ell) - u_\epsilon(\xi))d\xi \leq p(w_\epsilon(\theta)) - p(w_\epsilon(-\ell)). \tag{4.5}$$

Now since $w_\epsilon(\theta) > w_\epsilon(-\ell)$ we know the right hand side of (4.5) is bounded from above by $p(\beta) - p(\alpha)$. So for any $(\sigma_1, \sigma_2) \subset (-\ell, \theta)$ we have

$$\int_{\sigma_1}^{\sigma_2} (u_\epsilon(-\ell) - u_\epsilon(\xi))d\xi \leq p(\beta) - p(\alpha) \tag{4.6}$$

and hence

$$u_\epsilon(-\ell)(\sigma_2 - \sigma_1) - (p(\beta) - p(\alpha)) \leq \int_{\sigma_1}^{\sigma_2} u_\epsilon(\xi) d\xi.$$

Letting $\ell \to +\infty$ we find

$$\overline{u}(\sigma_2 - \sigma_1) - (p(\beta) - p(\alpha)) \leq \int_{\sigma_1}^{\sigma_2} u_\epsilon(\xi) d\xi. \tag{4.7}$$

If $(\sigma_1, \sigma_2) \subset (\theta, \ell)$ then $u_\epsilon(\xi) \geq u_\epsilon(-\ell)$ and we see

$$\overline{u}(\sigma_2 - \sigma_1) \leq \int_{\sigma_1}^{\sigma_2} u_\epsilon(\xi) d\xi. \tag{4.8}$$

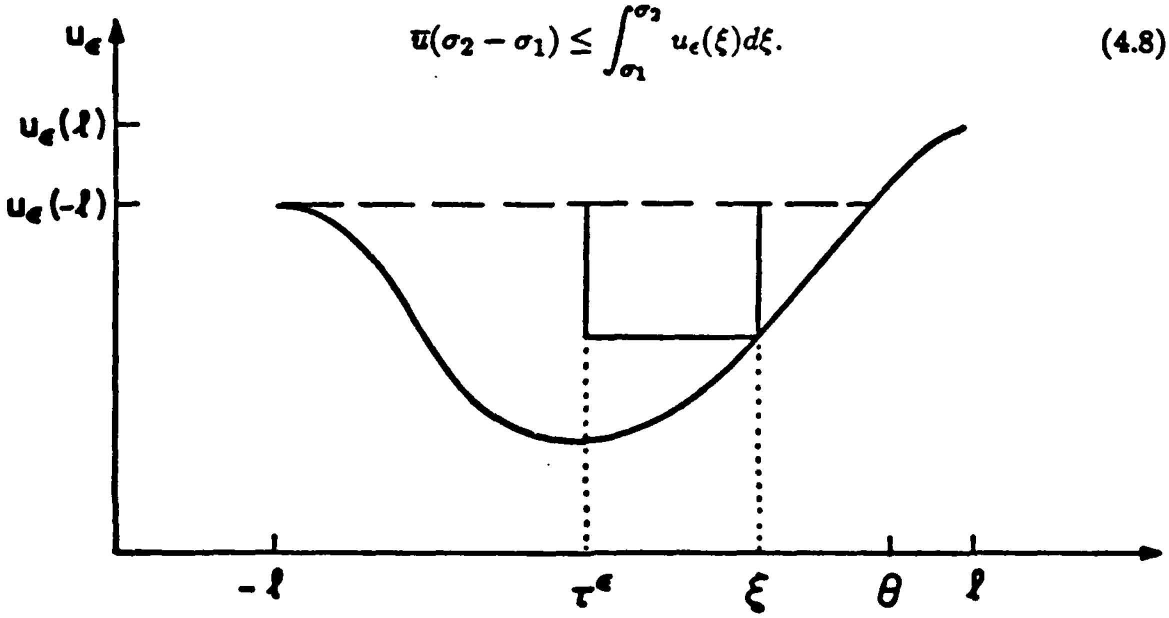


Figure 4.

Finally if $-\ell < \sigma_1 < \theta$, $\theta < \sigma_2 < \ell$ we write

$$\int_{\sigma_1}^{\sigma_2} u_\epsilon(\xi) d\xi = \int_{\sigma_1}^{\theta} u_\epsilon(\xi) d\xi + \int_{\theta}^{\sigma_2} u_\epsilon(\xi) d\xi$$

and use (4.7), (4.8) to again obtain (4.1).

To get the bound from below in (4.2) we observe from Figure 4 that when $\tau^\epsilon < \xi < \theta$ we have

$$(u_\epsilon(-\ell) - u(\xi))(\xi - \tau^\epsilon) \leq \int_{-\ell}^{\theta} (u_\epsilon(-\ell) - u_\epsilon(\xi)) d\xi. \tag{4.9}$$

From (4.9) and (4.6) we see

$$(u_\epsilon(-\ell) - u_\epsilon(\xi))(\xi - \tau^\epsilon) \leq p(\beta) - p(\alpha). \tag{4.10}$$

Now letting $\ell \to \infty$ we obtain (4.2). If $-\ell < \xi < \tau^\epsilon$ we again obtain (4.2) and of course if $\theta \le \xi \le \ell$ we trivially get (4.2). The proof for $u_\epsilon(-\ell) > u_\epsilon(\ell)$ is analogous.

Without going in details we sketch the appropriate constructions for ii(a), iii(a) if $u_\epsilon(-\ell) \le u_\epsilon(\ell)$ in Figure 5. Cases ii(b), ii(c) are done like i(a). Cases iii(b), (c), (d) are done like ii(a). In all cases crucial to the argument is the fact that $w_\epsilon(\theta) > w_\epsilon(\xi)$ so that $p(w_\epsilon(\theta)) - p(w_\epsilon(\zeta)) \le p(\beta) - p(\alpha)$ and inequality (4.6) is obtained without any monotonicity restrictions on $w_\epsilon(\xi)$ (in contrast to Thm. 2.3 of [8]).

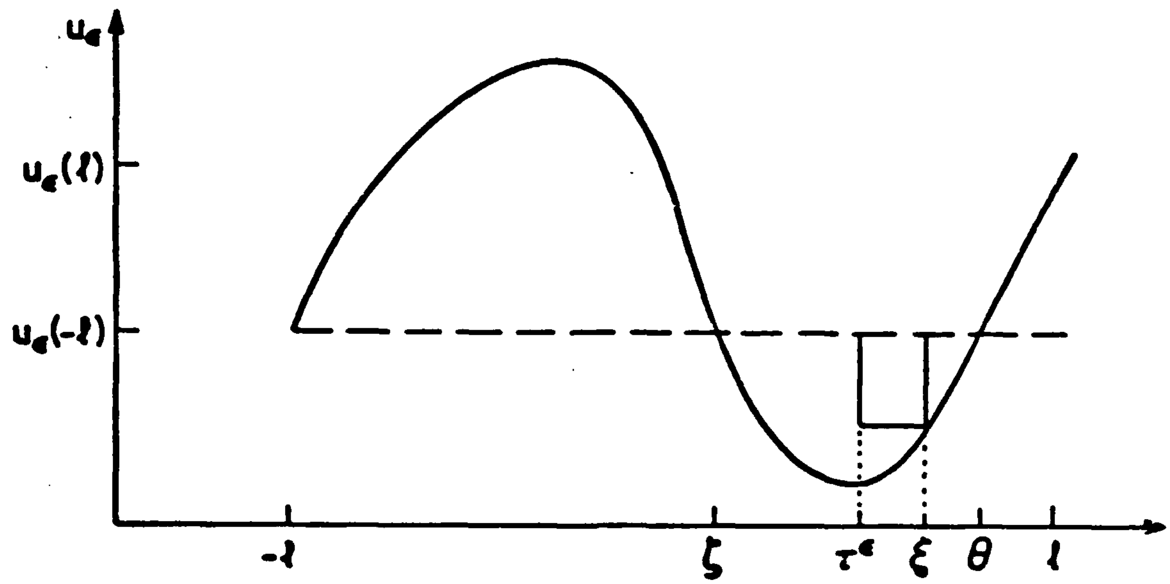


Figure 5. Case ii(a). Integrate (4.3) from ζ to θ.

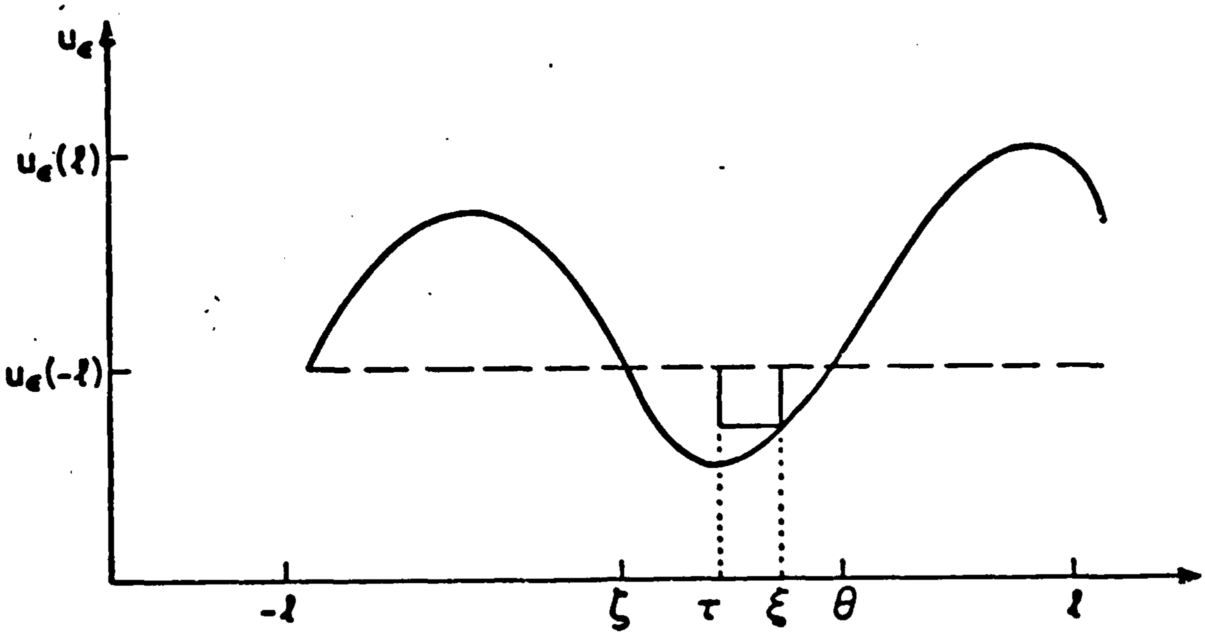


Figure 6. Case iii(a). Integrate (4.3) from ζ to θ.

This completes the proof of the lemma.

Lemma 4.2 Let $\{u_\epsilon(\xi), w_\epsilon(\xi);\ 0 < \epsilon < 1\}$ be a solution of P_ϵ as given by Corollary 2.7 when $w_- < a$, $w_+ > \beta$. Then for any given compact subset S of $(-\infty, 0)$ or $(0, \infty)$ there exists constants K and ϵ_0 (depending at most on u_-, u_+, w_-, w_+, p, S) so that

$$\sup_{\xi \in S}(|u_\epsilon(\xi)| + |w_\epsilon(\xi)|) \leq K \qquad \text{for} \quad 0 < \epsilon < \epsilon_0.$$

Proof. Let $S_+ \subset [a, b]$, $S_- \subset [-b, -a]$, $0 < a < b < \infty$. Then for ϵ sufficiently small $|\tau^\epsilon| \leq a/2$ and (4.2) yield $\sup_{\xi \in S_\pm} |u_\epsilon(\xi)| \leq K$. We now need to get a similar estimate on $w_\epsilon(\xi)$. In Cases i(a), (b) of Lemma 2.4, the proof of Lemma 3.5, 3.6, 3.7 yields a uniform in ϵ and ξ, $(-\infty < \xi < \infty)$, bound on $w_\epsilon(\xi)$ where as in Cases 0, i(c), ii(a), iii(a) $w_\epsilon(\xi)$ is monotone so that trivially $w_- \leq w_\epsilon(\xi) \leq w_+$ for $\xi \in (-\infty, \infty)$. Hence the only cases left to check are ii(b), ii(c), iii(b), iii(c), iii(d).

Case ii(b). On S_+, $w_\epsilon(\xi)$ is uniformly bounded in ϵ, ξ so we need only check S_-. Let $\eta \in S_-$, $\zeta \in S_+$. For ϵ sufficiently small $\eta < \tau_\epsilon < \zeta$. Integrate (0.10) from η to ζ to obtain

$$\epsilon u_\epsilon'(\zeta) - \epsilon u_\epsilon'(\eta) + \int_\eta^\zeta \xi u_\epsilon'(\xi) d\xi = p(w_\epsilon(\zeta)) - p(w_\epsilon(\eta)). \tag{4.10}$$

Since $u_\epsilon'(\zeta) > 0$, $u_\epsilon'(\eta) < 0$ (4.10) implies

$$\int_\eta^\zeta \xi u_\epsilon'(\xi) d\xi \leq p(w_\epsilon(\zeta)) - p(w_\epsilon(\eta))$$

and integration by parts yields

$$\zeta u_\epsilon(\zeta) - \eta u_\epsilon(\eta) - \int_\eta^\zeta u_\epsilon(\xi)d\xi \leq p(w_\epsilon(\zeta)) - p(w_\epsilon(\eta)). \tag{4.11}$$

Now use (4.1), (4.2) to bound the left hand side of (4.11) from below

$$\zeta\overline{u} - \frac{\zeta(p(\beta) - p(\alpha))}{|\zeta - \tau^\epsilon|} - \eta N_0 - N_0(\zeta - \eta) \leq p(w_\epsilon(\zeta)) - p(w_\epsilon(\eta)). \tag{4.12}$$

Since $\alpha \leq w_\epsilon(\zeta) \leq w_+$ we see $p(w_\epsilon(\zeta)) \leq p(\beta)$. Hence combined with this fact and $|\zeta - \tau^\epsilon| \geq a/2$ we see that (4.12) yields

$$-b|\overline{u}| - \frac{2b(p(\beta) - p(\alpha))}{a} - bN_0 - p(\beta) \leq -p(w_\epsilon(\eta)) \tag{4.13}$$

Since $w_\epsilon(\eta) \leq \beta$, (4.13) and the fact that $p(w) \to +\infty$ as $w \to -\infty$ show $w_\epsilon(\eta)$ uniformly bounded in ϵ, η for ϵ sufficiently small, $\eta \in S_-$.

Cases ii(c), iii(b). Proceed in a similar fashion as Case ii(b).

Case iii(c). From the mean value theorem there is $\zeta \in [1,2]$ so that $u'_\epsilon(\zeta) = u_\epsilon(2) - u_\epsilon(1)$ so by (4.2) $\epsilon u'_\epsilon(\zeta)$ is uniformly bounded. So for this ζ and arbitrary $\eta \in S_-$ we again derive (4.10) and since $u'_\epsilon(\eta) < 0$ we find

$$\begin{aligned} \epsilon u'_\epsilon(\zeta) - \int_\eta^\zeta u_\epsilon(\xi)d\xi &\leq p(w_\epsilon(\zeta)) - p(w_\epsilon(\eta)) \\ &\leq p(\beta) - p(w_\epsilon(\eta)). \end{aligned}$$

The same argument as given above for case ii(b) shows $w_\epsilon(\eta)$ uniformly bounded in ϵ, η for ϵ sufficiently small, $\eta \in S_-$.

Case iii(d). Proceed analogously as in Case iii(c).

Another set of bounds on $(u_\epsilon(\xi), w_\epsilon(\xi))$ is provided by the next lemma.

Lemma 4.3 Let $\{u_\epsilon(\xi), w_\epsilon(\xi); 0 < \epsilon < 1\}$ be a solution of P_ϵ as given by Corollary 2.7 when $w_- < \alpha$, $w_+ > \beta$. Let σ_-^ϵ, σ_+^ϵ denote the points of local minima and maxima for $w_\epsilon(\xi)$ and τ^ϵ the point of local minima for $u_\epsilon(\xi)$ (when they exist). Define $\overline{u} = \min(u_-, u_+)$,

$$B_\epsilon^- = \left(-\frac{1}{p'(w_-)}\right)^{1/2}\left(\overline{u} - \frac{(p(\beta) - p(\alpha))}{|\sigma_-^\epsilon - \tau^\epsilon|}\right) + w_- - \left(-\frac{1}{p'(w_-)}\right)^{1/2} u_-$$

$$B_\epsilon^+ = \left(-\frac{1}{p'(w_+)}\right)^{1/2}\left(\overline{u} + \frac{(p(\beta)-p(\alpha))}{|\sigma_+^\epsilon - \tau^\epsilon|}\right) + w_+ + \left(-\frac{1}{p'(w_+)}\right)^{1/2} u_+ .$$

Then in cases of Lemma 2.4 (with $\mu = 1$, $L = \infty$) we have the following estimates:

In cases 0, i(a,b,c), (3.1) holds.

In the rest of the cases $u_\epsilon(\xi)$ satisfies (4.2) and $w_\epsilon(\xi)$ satisfies

$w_- \le w_\epsilon(\xi) \le w_+$ in Cases i(d), ii(a), iii(a);
$B_\epsilon^- \le w_\epsilon(\xi) \le w_+$ in Cases ii(b), iii(c);
$w_- \le w_\epsilon(\xi) \le B_\epsilon^+$ in Cases ii(c), iii(d);
$B_\epsilon^- \le w_\epsilon(\xi) \le B_\epsilon^+$ in Case iii(b).

Proof. For Cases 0, i(a,b,c) the result was given in Lemma 3.5 and Cases i(d), ii(a), iii(a) are trivial.

For Case ii(b) follow the method of proof of Case 1(a), Lemma 3.5. Upon reaching inequality (3.3) replace $u_\epsilon(\sigma_-^\epsilon)$ by the bound from below given by (4.2). Similarly in Case ii(c) obtain the bound for $w_\epsilon(\sigma_+^\epsilon)$:

$$w_+ - w_\epsilon(\sigma_+^\epsilon) \le \left(-\frac{1}{p'(w_+)}\right)(u_+ - u_\epsilon(\sigma_+^\epsilon))$$

and again bound $u_\epsilon(\sigma_+^\epsilon)$ using (4.2). Case iii(c) follows like ii(b), Case iii(d) follows like ii(c), and Case iii(b) uses *the arguments of both* ii(b) and ii(c).

We now combine Lemmas 4.2 and 4.3 to get the following improved estimate on $\{u_\epsilon(\xi), w_\epsilon(\xi); 0 < \epsilon < 1\}$.

Lemma 4.4 Let $\{u_\epsilon(\xi), w_\epsilon(\xi); 0 < \epsilon < 1\}$ be a solution of P_ϵ as given by Corollary 2.7 when $w_- < \alpha$, $w_+ > \beta$. Then on any semi-infinite interval $(-\infty, -a]$ or $[a, \infty)$, $a > 0$ there exists constants k and ϵ_0 (depending at most on u_-, u_+, w_-, w_+, p, a) so that

$$\begin{aligned} \sup_{(-\infty,a]} (|u_\epsilon(\xi)| + |w_\epsilon(\xi)|) \le k, \\ \sup_{[a,\infty)} (|u_\epsilon(\xi)| + |w_\epsilon(\xi)|) \le k, \end{aligned} \tag{4.14}$$

for $0 < \epsilon < \epsilon_0$.

Proof. The bounds on $u_\epsilon(\xi)$, $w_\epsilon(\xi)$ in Cases 0, i(a,b,c) are known from inequality (3.1). For the rest of the cases the bounds on $u_\epsilon(\xi)$ follow from inequality (4.2). So we only have to produce the relevant bounds on $w_\epsilon(\xi)$. Here again Lemma 4.3 tells us Case i(d), ii(a), iii(a) are trivial so let us move on to Case ii(b). In Case ii(b) we know from Lemma 4.3 that $B_\epsilon^- \le w_\epsilon(\xi) \le w_+$. Since $\tau^\epsilon \to 0$ as $\epsilon \to 0+$,

B_ϵ^- will be bounded from below for sufficiently small ϵ if $\sigma^\epsilon \not\to 0$ as $\epsilon \to 0+$. On the other hand if $\sigma^\epsilon \to 0$ as $\epsilon \to 0+$ it means the minimum of w_ϵ on $(-\infty,\infty)$ is taken on at σ_-^ϵ where $|\sigma_-^\epsilon| \leq C$, a constant. If $-C \leq \sigma_-^\epsilon \leq -a$ then Lemma 4.3 tells us $w_\epsilon(\sigma_-^\epsilon)$ is bounded from below. If $-a < \sigma_-^\epsilon \leq 0$ then the minimum of w_ϵ on $(-\infty,-a]$ is taken on at $\xi = -a$. By Lemma 4.2 we know $w_\epsilon(-a)$ is uniformly bounded from below. and so we see $w_\epsilon(\xi)$ has a uniform (in ξ,ϵ) bound from below on $(-\infty,-a]$. The other cases, i.e. ii(c), iii(b), (c), (d) follow similarly.

Lemma 4.5 Let $\{u_\epsilon(\xi), w_\epsilon(\xi), 0 < \epsilon < 1\}$ be a solution of P_ϵ as given by Corollary 2.7 when $w_- < \alpha$, $w_+ > \beta$. Then sequence $(u_\epsilon(\xi), w_\epsilon(\xi))$ possesses a subsequence which converges a.e. on $(-\infty,\infty)$ to functions $(u(\xi), w(\xi))$. On compact subsets of $(-\infty,0)\cup(0,\infty)$ the convergent subsequence is bounded uniformly in ϵ with uniformly bounded total variation. The limit functions have bounded variation on compact subsets of $(-\infty,0)\cup(0,\infty)$.

Proof. On the finite domain $[-1,-1/2]\cup[1/2,2]$ $(= R_2)$ Lemma 2.4 (with $\mu = 1, L = \infty$) and Lemma 4.4 combined with Helly's theorem ([10], p. 222) provides a convergent subsequence $\{(u_{\epsilon_2}(\xi), w_{\epsilon_2}(\xi))\}$ which converges boundly to functions $u(\xi), w(\xi)$ defined on R_2; $u(\xi)$, $w(\xi)$ have bounded variation R_2. Now consider $\{u_{\epsilon_2}(\xi), w_{\epsilon_2}(\xi)\}$ on $[-3,-1/3]\cup[1/3,3]$ $(= R_3)$. Again we extract a convergent subsequence which converges boundedly to the functions $u(\xi)$, $w(\xi)$ on R_2 and extensions of $u(\xi)$, $w(\xi)$ (also denoted by the same letters) on R_3. Continue this process on each R_n, $n = 2,3,4,\ldots$ Finally extract the diagonal element of each enumerated sequence. The sequence of diagonal elements is convergent at each $\xi \neq 0$ to functions $u(\xi)$, $w(\xi)$ defined on $(-\infty,0)\cup(0,\infty)$. The remarks regarding compact subsets of $(-\infty,0)\cup(0,\infty)$ follow directly from Helly's Theorem.

Lemma 4.6 The functions $u(\xi)$, $w(\xi)$ defined by Lemma 4.5 satisfy the boundary conditions

$$\begin{aligned} u(-\infty) = u_- \quad &, \quad u(-\infty) = u_+, \\ w(-\infty) = w_- \quad &, \quad w(+\infty) = w_+. \end{aligned}$$

Proof. We follow the proof of Thm. 3.2 of [7]. Let $\mathbf{y}_\epsilon(\xi) = \binom{u_\epsilon(\xi)}{w_\epsilon(\xi)}$, $\mathbf{f}(\mathbf{y}_\epsilon) = \binom{p(w_\epsilon)}{-u_\epsilon}$. Then (0.10), (0.11) imply

$$\frac{d}{d\xi}(\exp(\xi^2/2\epsilon)\mathbf{y}_\epsilon'(\xi)) = \frac{1}{\epsilon}(\nabla \mathbf{f}(\mathbf{y}_\epsilon)\mathbf{y}_\epsilon'(\xi)\exp(\frac{\xi^2}{2\epsilon}))$$

and integrating from 1 to ξ, $\xi > 1$, we find

$$\exp(\xi^2/2\epsilon)\mathbf{y}_\epsilon'(\xi) - \exp(\frac{1}{2\epsilon})\mathbf{y}_\epsilon'(1) = \frac{1}{\epsilon}\int_1^\xi \nabla\mathbf{f}(\mathbf{y}_\epsilon)\mathbf{y}_\epsilon'(\zeta)\exp(\frac{\zeta^2}{2\epsilon})d\zeta.$$

Since by Lemma 4.4 $|\mathbf{y}_\epsilon(\xi)|$ is uniformly bounded by k on $[1,\infty)$ we know $|\nabla \mathbf{f}(\mathbf{y}_\epsilon)| \leq R$ for some constant $R > 0$. Thus

$$|\exp(\xi^2/2\epsilon)\mathbf{y}_\epsilon'(\xi)| \leq |\exp(\frac{1}{2\epsilon})\mathbf{y}_\epsilon'(1)| + \frac{R}{\epsilon}\int_1^\xi |\mathbf{y}_\epsilon'(\zeta)|\exp(\frac{\zeta^2}{2\epsilon})d\zeta$$

and using Gronwall's inequality we find

$$|\exp(\xi^2/2\epsilon)\mathbf{y}_\epsilon'(\xi)| \leq |\exp(\frac{1}{2\epsilon})\mathbf{y}_\epsilon'(1)|\exp\left[\frac{R}{\epsilon}(\xi-1)\right]$$

and hence

$$|\mathbf{y}_\epsilon'(\xi)| \leq |\mathbf{y}_\epsilon'(1)|\exp\left(\frac{2R\xi - 2R + 1 - \xi^2}{2\epsilon}\right). \tag{4.15}$$

Now note that

$$\begin{aligned}
\exp(\xi^2/2\epsilon)\mathbf{y}_\epsilon'(\xi) &= \mathbf{z}_1 + \frac{1}{\epsilon}\int_1^\xi \mathbf{f}(\mathbf{y}_\epsilon(\zeta))'\exp(\frac{\zeta^2}{2\epsilon})d\zeta \\
&= \mathbf{z}_1 + \frac{1}{\epsilon}\mathbf{f}(\mathbf{y}_\epsilon(\xi))\exp(\frac{\xi^2}{2\epsilon}) - \frac{1}{\epsilon}\mathbf{f}(\mathbf{y}_\epsilon(1))\exp(\frac{1}{2\epsilon}) \\
&\quad - \frac{1}{\epsilon^2}\int_1^\xi \zeta\mathbf{f}(\mathbf{y}_\epsilon(\zeta))\exp(\frac{\zeta^2}{2\epsilon})d\zeta \\
&= \mathbf{z}_2 + \frac{1}{\epsilon}\mathbf{f}(\mathbf{y}_\epsilon(\xi))\exp(\frac{\xi^2}{2\epsilon}) - \frac{1}{\epsilon^2}\int_1^\xi \zeta\mathbf{f}(\mathbf{y}_\epsilon(\zeta))\exp(\frac{\zeta^2}{2\epsilon})d\zeta
\end{aligned}$$

and hence

$$\mathbf{y}_\epsilon'(\xi) = \mathbf{z}_2\exp\left(\frac{-\xi^2}{2\epsilon}\right) + \frac{1}{\epsilon}\mathbf{f}(\mathbf{y}_\epsilon(\xi)) - \frac{1}{\epsilon^2}\int_1^\xi \zeta\mathbf{f}(\mathbf{y}_\epsilon(\zeta))\exp\left(\frac{\zeta^2}{2\epsilon}\right)d\zeta. \tag{4.16}$$

Here

$$\begin{aligned}
\mathbf{z}_2\int_1^2 \exp(-\xi^2/2\epsilon)d\xi &= \mathbf{y}_\epsilon(2) - \mathbf{y}_\epsilon(1) - \frac{1}{\epsilon}\int_1^2 \mathbf{f}(\mathbf{y}_\epsilon(\xi))d\xi \\
&\quad + \frac{1}{\epsilon^2}\int_1^2 \zeta\mathbf{f}(\mathbf{y}_\epsilon(\zeta))\exp(\frac{\zeta^2}{2\epsilon})d\zeta.
\end{aligned} \tag{4.17}$$

So from (4.16) we see

$$\begin{aligned}
|\mathbf{y}_\epsilon'(1)| &\leq |\mathbf{z}_2|\exp(-\frac{1}{2\epsilon}) + \frac{1}{\epsilon}|\mathbf{f}(\mathbf{y}_\epsilon(1))| \\
&\leq |\mathbf{z}_2|\exp(-\frac{1}{2\epsilon}) + \frac{\text{const.}}{\epsilon}
\end{aligned} \tag{4.18}$$

From (4.17) and the inequality

$$\int_1^2 \exp\left(-\frac{\xi^2}{2\epsilon}\right)d\xi \geq \exp\left(-\frac{2}{\epsilon}\right)$$

we see

$$|z_2| \leq (\text{const.} + \frac{\text{const.}}{\epsilon} + \frac{\text{const.}}{\epsilon^2}\exp(\frac{2}{\epsilon}))\exp(\frac{2}{\epsilon})$$

and hence by (4.18) that

$$|\mathbf{y}'_\epsilon(1)| \leq \frac{\text{const.}}{\epsilon^2}\exp(\frac{7}{2\epsilon}). \tag{4.19}$$

Now insert (4.19) into (4.15) to find

$$|\mathbf{y}'_\epsilon(\xi)| \leq \frac{\text{const.}}{\epsilon^2}\left(\frac{2R\xi - 2R + 8 - \xi^2}{2\epsilon}\right). \tag{4.20}$$

So for $\xi > R+(R^2-2R+8)^{1/2}$ (4.20) shows that $|\mathbf{y}'_\epsilon(\xi)| \to 0$ as $\epsilon \to 0+$. Recalling that $(u_\epsilon(\xi),\ w_\epsilon(\xi))$ converge pointwise to $u(\xi)$, $w(\xi)$ we see $u(\xi)$, $w(\xi)$ must be constants for $\xi > R + (R^2 - 2R + 8)^{1/2}$. Since for any $\epsilon > 0$ $\lim_{\xi\to\infty} u_\epsilon(\xi) = u_+$, $\lim_{\xi\to\infty} w_\epsilon(\xi) = w_+$ these constants must u_+, w_+. A similar argument works for $\xi = -\infty$.

Corollary 4.7 The functions $u(\xi)$, $w(\xi)$ defined by Lemma 4.5 satisfy the conditions

$$\left.\begin{array}{l} u(\xi) = u_-, \\ w(\xi) = w_-, \end{array}\right. \quad \xi < -M, \qquad \left.\begin{array}{l} u(\xi) = u_+ \\ w(\xi) = w_+ \end{array}\right. \quad \xi > M$$

for some positive constant M.

Lemma 4.8 The functions $u(\xi)$, $w(\xi)$ defined by Lemma 4.5 satisfy

$$\begin{aligned} p(w)' - \xi u' &= 0 \\ -u' - \xi w' &= 0 \end{aligned} \tag{4.21}$$

in the sense of distributions at any $\xi \neq 0$.

At any point $\xi_0 \neq 0$ of discontinuity of $u(\xi)$, $w(\xi)$ the Rankine-Hugoniot jump conditions are satisfied

$$\begin{aligned} p(w(\xi_0+)) - p(w(\xi_0-)) - \xi_0(u(\xi_0+) - u(\xi_0-)) &= 0, \\ -(u(\xi_0+) - u(\xi_0-)) - \xi_0(w(\xi_0+) - w(\xi_0-)) &= 0. \end{aligned} \tag{4.22}$$

Proof. On any compact subset of $(0,\infty)$ or $(-\infty,0)$ we have a sequence of solutions of (0.10), (0.11) which converges boundedly a.e. Hence if we multiply (0.10), (0.11) by C^∞ test functions with compact support excluding $\xi = 0$, integrate

by parts, pass to the limits as the relevant sequence of ϵ's goes to zero, and use the Lebesgue dominated convergence theorem we obtain (4.21). Equation (4.22) follows from (4.21) in the standard manner (e.g. [7], (3.14)).

Corollary 4.7 and Lemma 4.8 take us very close to asserting solvability of the Riemann problem. Unfortunately we still must deal with the behavior of u, w at the troublesome point $\xi = 0$, i.e. we have yet to show u, w solve (4.21) in a neighborhood of $\xi = 0$. In fact the derivation of the Rankine-Hugoniot jump condition for weak solutions [8] shows that u, w will be a distributional solution of (4.21) at $\xi = 0$ if

$$\begin{aligned} \lim_{\xi \to 0-} p(w(\xi)) &= \lim_{\xi \to 0+} p(w(\xi)), \\ \lim_{\xi \to 0-} u(\xi) &= \lim_{\xi \to 0+} u(\xi), \end{aligned} \tag{4.23}$$

where the indicated limits exist (finite).

Before pursuing the study of (4.23) we first show that $u(\xi)$, $w(\xi)$ are locally integrable. First we make

Assumption 4.9 Assume

$$\frac{|\int_\beta^w p(\zeta)d\zeta|}{|w|} \to \infty \qquad \text{as } |w| \to \infty.$$

Lemma 4.10 If Assumption 4.9 holds then $\{w_\epsilon(\xi)\}$ has absolutely equicontinuous integrals and the functions $u(\xi)$, $w(\xi)$ defined by Lemma 4.5 are locally integrable in $(-\infty, \infty)$.

Proof. First we know from Lemma 4.1 (4.1) that $|u_\epsilon(\xi)|$ is locally integrable. Since a subsequence of $u_\epsilon(\xi)$ converges to $u(\xi)$, Fatou's theorem implies the limit function $u(\xi)$ is locally integrable. To show local integrability of $w(\xi)$ we proceed indirectly. A theorem of D. Vitali ([11], p. 152) tells us that if $\{w_\epsilon(\xi)\}$ have absolutely equicontinuous integrals, then $w(\xi)$ is locally integrable and moreover

$$\lim_{\epsilon_n \to 0} \int_{\sigma_1}^{\sigma_2} w_{\epsilon_n}(\xi)d\zeta = \int_{\sigma_1}^{\sigma_2} w(\xi)d\xi. \tag{4.24}$$

Here the limit is taken on the a.e. convergent subsequence of $w_\epsilon(\xi)$ denoted as $w_{\epsilon_n}(\xi)$. Recall that to have an absolutely equicontinuous integral we need for every $\delta > 0$ there is a $\ell(\delta)$ such that if $0 < \sigma_2 - \sigma_1 \leq \ell(\delta)$

$$|\int_{\sigma_1}^{\sigma_2} w_\epsilon(\xi)d\xi| < \delta \qquad \text{for all } \epsilon > 0.$$

We now establish absolute equicontinuity of the integral of $w_\epsilon(\xi)$ by using the argument of Lemma 3.4 of [8] (which is itself a variant of the test of de là Valée-Poussin [11], p. 154.)

First notice however that in i(d), ii(a), iii(a) of Lemma 2.4 there is nothing to prove since $w_\epsilon(\xi)$ is monotone and hence uniformly bounded in ξ, ϵ. Also recall Cases 0, i(a), (b), (c) were covered by Theorem 3.8. So we need only consider Cases ii(b), (c), iii(b), (c), (d).

Let us consider Case ii(c). Given any interval (ℓ_1, ℓ_2) we either (I) divide it into the subintervals $(\ell_1, t_\epsilon]$ where $w_- \le w_\epsilon(\xi) \le \beta$ and $[t_\epsilon, \ell_2)$ where $\beta \le w_\epsilon(\xi)$, $w_\epsilon(t_\epsilon) = \beta$, (II) $w_\epsilon(\xi) \ge \beta$ on (ℓ_1, ℓ_2) or (III) $w_\epsilon(\xi) \le \beta$ on (ℓ_1, ℓ_2).

First we consider possibility (I).

Multiply (0.10) by u_ϵ and (0.11) by $-p(w)$ and add. If we define $\eta(u,w) = \frac{u^2}{2} - \int_\beta^w p(s)d\zeta$ and $\eta_\epsilon(\xi) = \eta(u_\epsilon(\xi), w_\epsilon(\xi))$ we see

$$\epsilon\eta_\epsilon''(\xi) + \xi\eta_\epsilon'(\xi) - (u_\epsilon(\xi)p(w_\epsilon(\xi)))' - \epsilon u_\epsilon'(\xi)^2 + \epsilon p'(w_\epsilon(\xi))w_\epsilon'(\xi)^2 = 0. \tag{4.25}$$

Let $\overline{\eta} = \max(\eta(u_-, w_-), \eta(u_+, w_+))$. On any sub-interval $(\sigma_1', \sigma_2') \subset [t_\epsilon, \ell_2)$ if $\eta_\epsilon(\sigma_1') > \overline{\eta}$ set $\zeta_\epsilon = \sup\{\xi \in [t_\epsilon, \sigma_1']; \eta_\epsilon(\xi) \le \overline{\eta}\}$. If $\eta_\epsilon(\sigma_1') \le \overline{\eta}$ set $\zeta_\epsilon = \inf\{\xi \in (\sigma_1', \sigma_2'); \eta_\epsilon(\xi) \ge \overline{\eta}\}$ if this set is not empty. Similarly if $\eta_\epsilon(\sigma_2') > \overline{\eta}$ set $\theta_\epsilon = \inf\{\xi \in (\sigma_2', \ell_2]; \eta_\epsilon(\xi) \le \overline{\eta}\}$ while if $\eta_\epsilon(\sigma_2') \le \overline{\eta}$ set $\theta_\epsilon = \sup\{\xi \in (\sigma_1', \sigma_2'); \eta_\epsilon(\xi) \ge \overline{\eta}\}$. Observe that $\eta_\epsilon'(\zeta_\epsilon) \ge 0$ and $\eta_\epsilon'(\theta_\epsilon) \le 0$, and

$$\begin{aligned}\int_{\sigma_1'}^{\sigma_2'} (\eta_\epsilon(\xi) - \overline{\eta})d\xi &\le \int_{\zeta_\epsilon}^{\theta_\epsilon} (\eta_\epsilon(\xi) - \overline{\eta}d\xi \\ &= -\int_{\zeta_\epsilon}^{\sigma_\epsilon} \xi\eta_\epsilon'(\xi)d\xi.\end{aligned} \tag{4.26}$$

So if we integrate (4.25) over $(\zeta_\epsilon, \sigma_\epsilon)$ and use (4.26) we see

$$\begin{aligned}\int_{\sigma_1'}^{\sigma_2'} (\eta_\epsilon(\xi) - \overline{\eta})d\xi + \epsilon\int_{\zeta_\epsilon}^{\theta_\epsilon} (u_\epsilon'(\xi)^2 - p'(w_\epsilon(\xi))w_\epsilon'(\xi)^2)d\xi \\ \le -u_\epsilon(\theta_\epsilon)p(w_\epsilon(\theta_\epsilon)) + u_\epsilon(\zeta_\epsilon)p(w_\epsilon(\zeta_\epsilon)).\end{aligned} \tag{4.27}$$

By the definitions of θ_ϵ, ζ_ϵ we see $\eta(u_\epsilon(\theta_\epsilon), w_\epsilon(\theta_\epsilon))$ and $\eta(u_\epsilon(\zeta_\epsilon), w_\epsilon(\zeta_\epsilon))$ are uniformly bounded from above and since $w_\epsilon(\theta_\epsilon), w_\epsilon(\zeta_\epsilon)$ are greater than β, η is convex at these values. This implies $u_\epsilon(\theta_\epsilon)$, $w_\epsilon(\theta_\epsilon)$, $u_\epsilon(\zeta_\epsilon)$, $w_\epsilon(\zeta_\epsilon)$ are uniformly bounded in ϵ. Hence the right hand side of (4.27) is bounded by a constant K independent of ϵ.

Now since $-\frac{1}{w}\int_\beta^w p(\delta)ds \to \infty$ as $w \to \infty$ for arbitrary $\delta > 0$ there is a $w_0 \ge \beta$ so that

$$\frac{w}{\eta(u,w)} < \frac{\delta}{2K} \qquad \text{for all } w \ge w_0.$$

We then set $\ell(\delta) = \frac{\delta}{(|w_-|+\beta+w_0+\overline{\eta}\frac{\delta}{2K})}$. Fix σ_1, σ_2, $0 < \sigma_2 - \sigma_1 < \ell(\delta)$. We note for any σ_1, σ_2, $\sigma_1 \in (\ell_1, t_\epsilon]$, $\sigma_2 \in (t_\epsilon, \ell_2]$

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} w_\epsilon(\xi)d\xi &= \int_{\sigma_1}^{t_\epsilon} w_\epsilon(\xi)d\xi + \int_{t_\epsilon}^{\sigma_2} w_\epsilon(\xi)d\xi \\ &\le \beta(t_\epsilon - \sigma_1) + \int_{t_\epsilon}^{\sigma_2}(w_0 + \frac{\delta}{2K}\eta(u_\epsilon(\xi), w_\epsilon(\xi)))d\xi \\ &\le \beta(t_\epsilon - \sigma_1) + (\sigma_2 - t_\epsilon)w_0 \\ &\quad + \frac{\delta}{2K}\int_{t_\epsilon}^{\sigma_2} \eta(u_\epsilon(\xi), w_\epsilon(\xi))d\xi.\end{aligned}$$

Now use (4.27) with $\sigma_2' = \sigma_2$, $\sigma_1' = t_\epsilon$ and we see

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} w_\epsilon(\xi)d\xi &\le \beta(t_\epsilon - \sigma_1) + (\sigma_2 - t_\epsilon)w_0 + \frac{\delta}{2K}(K + \overline{\eta}(\sigma_2 - \sigma_1)) \\ &\le (\sigma_2 - \sigma_1)(\beta + w_0 + \frac{\overline{\eta}\delta}{2K}) + \frac{\delta}{2} \le \delta.\end{aligned} \tag{4.28}$$

If $\sigma_1, \sigma_2 \ge t_\epsilon$

$$\int_{\sigma_1}^{\sigma_2} w_\epsilon(\xi)d\xi \le \int_{\sigma_1}^{\sigma_2}(w_0 + \frac{\delta}{2K}\eta(u_\epsilon(\xi), w_\epsilon(\xi)))d\xi \le \delta \tag{4.29}$$

and if $\sigma_1, \sigma_2 \le t_\epsilon$

$$\int_{\sigma_1}^{\sigma_2} w_\epsilon(\xi)d\xi \le \beta(\sigma_2 - \sigma_1) \le \delta. \tag{4.30}$$

Also since $w_\epsilon(\xi) \ge w_-$ we easily see

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} w_\epsilon(\xi)d\xi &\ge w_-(\sigma_2 - \sigma_1) \\ &\ge -|w_-|(\sigma_2 - \sigma_1) \ge -\delta.\end{aligned} \tag{4.31}$$

So we see for (I) that (4.28)-(4.31) imply

$$|\int_{\sigma_1}^{\sigma_2} w_\epsilon(\xi)d\xi| \le \delta \qquad \text{if } 0 < \sigma_2 - \sigma_1 < \ell(\delta).$$

A similar argument of course works for (II) while (III) is trivial. So in ii(c) we know $w_\epsilon(\xi)$ possesses absolutely equicontinuous integrals. Cases ii(b), iii(b), (c), (d) are done in a similar fashion. As advertised above Vitali's theorem thus tells us $w(\xi)$ is locally integrable and (4.24) holds.

The next Lemma follows almost identically from Lemma 3.3 of [S].

Lemma 4.11 The four limits which appear in (4.23) always exist (finite) and (4.23_1) is always satisfied. Equation (4.23_2) is satisfied if the sequence $\{\int_0^\xi u_\epsilon(\xi)d\xi\}$ (taken on the convergent subsequence of Lemma 4.5) is absolutely equicontinuous. Furthermore the relation $-(p(\beta)-p(\alpha)) \le \lim_{\xi\to 0_-} -p(w(\xi))+\lim_{\xi\to 0+} p(w(\xi)) \le 0$ holds in general.

Proof. Let $\{u_\epsilon(\xi), w_\epsilon(\xi)\}$ denote the convergent subsequence of Lemma 4.5. Note that since $w_\epsilon(\xi)$, $u_\epsilon(\xi)$ are piecewise monotone as $(-\infty,\infty)$ (Lemma 2.4 with $L=\infty$) then the limit functions $u(\xi), w(\xi)$ are also. Hence the set of points of continuity (in fact differentiability) of u, w is dense in any finite ξ interval.

Now let ζ and θ be points of continuity of $u(\xi), w(\xi)$, $\zeta < 0 < \theta$. From the mean value theorem for every small $\epsilon > 0$ we can find $\zeta_\epsilon \in \lfloor \zeta - \epsilon^{1/2}, \zeta]$, $\theta_\epsilon \in \lfloor \sigma, \theta^{1/2} + \epsilon^{1/2}]$ so that

$$\epsilon^{1/2}u_\epsilon'(\zeta_\epsilon) = u_\epsilon(\zeta) - u_\epsilon(\zeta - \epsilon^{1/2}), \quad \epsilon^{1/2}w_\epsilon'(\zeta_\epsilon) = w_\epsilon(\zeta) - w_\epsilon(\zeta - \epsilon^{1/2}),$$
$$\epsilon^{1/2}u_\epsilon'(\theta_\epsilon) = u_\epsilon(\theta + \epsilon^{1/2}) - u_\epsilon(\theta), \quad \epsilon^{1/2}w_\epsilon'(\theta_\epsilon) = w_\epsilon(\theta_\epsilon^{1/2} + \epsilon^{1/2}) - w_\epsilon(\theta).$$

By Lemma 2.4 there are constants K_θ, K_ζ so that

$$\begin{aligned} |\epsilon^{1/2}u_\epsilon'(\zeta_\epsilon)| \le K_\zeta, \quad |\epsilon^{1/2}w_\epsilon'(\zeta_\epsilon)| \le K_\zeta, \\ |\epsilon^{1/2}w_\epsilon'(\theta_\epsilon)| \le K_\theta, \quad |\epsilon^{1/2}w_\epsilon'(\theta_\epsilon)| \le K_\theta, \end{aligned} \tag{4.32}$$

for ϵ sufficiently small. (Of course K_θ, K_ζ maybe becomes unbounded as $\theta, \zeta \to 0$ but for the moment ζ, θ are fixed.)

Now we integrate (0.10), (0.11) on $(\zeta_\epsilon, \theta_\epsilon)$ obtaining

$$\begin{aligned} \epsilon u_\epsilon'(\theta_\epsilon) - \epsilon u_\epsilon'(\zeta_\epsilon) + \theta_\epsilon u_\epsilon(\theta_\epsilon) - \zeta_\epsilon u_\epsilon(\zeta_\epsilon) - \int_{\zeta_\epsilon}^{\theta_\epsilon} u_\epsilon(\xi)d\xi \\ = p(w_\epsilon(\theta_\epsilon)) - p(w_\epsilon(\zeta_\epsilon)), \end{aligned} \tag{4.33}$$

$$\begin{aligned} \epsilon w_\epsilon'(\theta_\epsilon) - \epsilon w_\epsilon'(\zeta_\epsilon) + \theta_\epsilon w_\epsilon(\theta_\epsilon) - \zeta_\epsilon w_\epsilon(\zeta_\epsilon) - \int_{\zeta_\epsilon}^{\theta_\epsilon} w_\epsilon(\xi)d\xi \\ = -u_\epsilon(\theta_\epsilon) + u_\epsilon(\zeta_\epsilon). \end{aligned} \tag{4.34}$$

Now let $\epsilon \to 0+$ in (4.33), (4.34). Since θ, ζ are points of continuity of u, w we find by virtue of (4.32) that

$$\theta u(\theta) - \zeta u(\zeta) - p(w(\theta)) + p(w(\zeta)) = \lim_{\epsilon\to 0+} \int_{\zeta_\epsilon}^{\theta_\epsilon} u_\epsilon(\xi)d\xi, \tag{4.35}$$

$$\theta w(\theta) - \zeta w(\zeta) + u(\theta) - u(\zeta) = \int_\zeta^\theta w(\xi)d\xi, \tag{4.36}$$

where in (4.36) we have used (4.24) i.e. the absolute equicontinuity of the integrals of $w_\epsilon(\xi)$. Notice the limit on the right hand side of (4.35) exists since the limits on the left hand side exist. From (4.1) we see that $\lim_{\epsilon\to 0+}\int_{\zeta_\epsilon}^{\theta_\epsilon} u_\epsilon(\xi)d\xi \overset{\text{def}}{=} S(\zeta,\theta)$ satisfies $\overline{u}(\zeta-\theta) - (p(\beta)-p(\alpha)) \le S(\zeta,\theta) \le N_0(\zeta-\theta)$. By Lemma 4.4 for fixed $\zeta < 0$, $S(\zeta,\theta)$ is continuous in θ, $\theta > 0$, $|\theta|$ small and for fixed $\theta > 0$, $S(\zeta,\theta)$ is continuous in ζ, $\zeta < 0$, $|\zeta|$ small.

Now since $|w(\xi)|$ may be infinite only at $\xi = 0$ (again by Lemma 4.4) pointwise limits of ii(b), (c), iii(b), (c), (d) of Lemma 2.4 shows that if $|w(0)| = \infty$, w must have one of three shapes shown in Figure 7.

In all three cases (I), (II), (III) we see

$$|\zeta w(\zeta)| \le \int_\zeta^\theta |w(\xi)|d\xi,$$

$$|\theta w(\theta)| \le \int_\zeta^\theta |w(\xi)|d\xi.$$

But since $w(\xi)$ is locally integrable (Lemma 4.10) we see

$$\lim_{\theta\to 0+}\theta w(\theta) = \lim_{\zeta\to 0-}\zeta w(\zeta) = \lim_{\substack{\theta\to 0+\\ \zeta\to 0-}}\int_\zeta^\theta w(\xi)d\xi = 0.$$

Since $u(\xi)$ has the shape of (I) near $\xi = 0$ and $u(\xi)$ is locally integrable (Lemma 4.10)

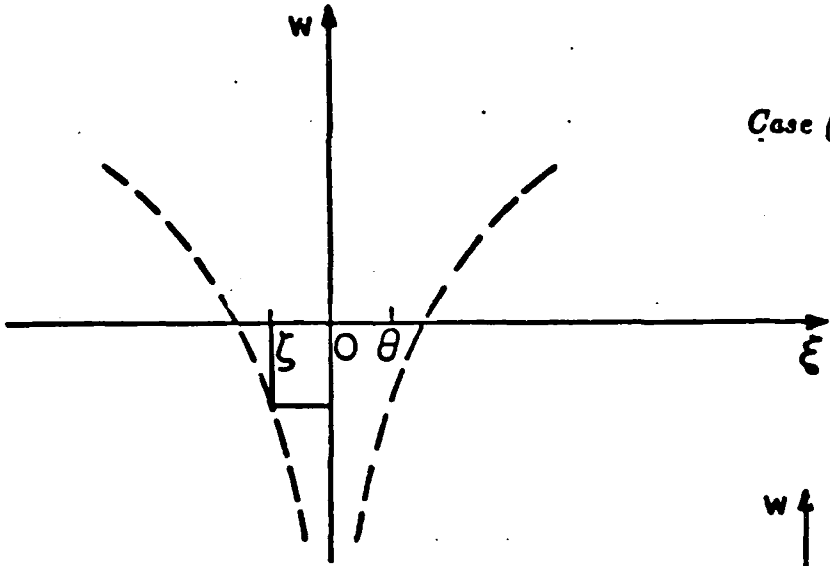
$$\lim_{\theta\to 0+}\theta u(\theta) = \lim_{\zeta\to 0-}\zeta u(\zeta) = 0.$$

Now let $\theta \to 0+$, $\zeta \to 0-$ along a sequence of points of continuity of u, w and possibly extracting a further subsequence so that $S(\zeta,\theta)$ converges we find

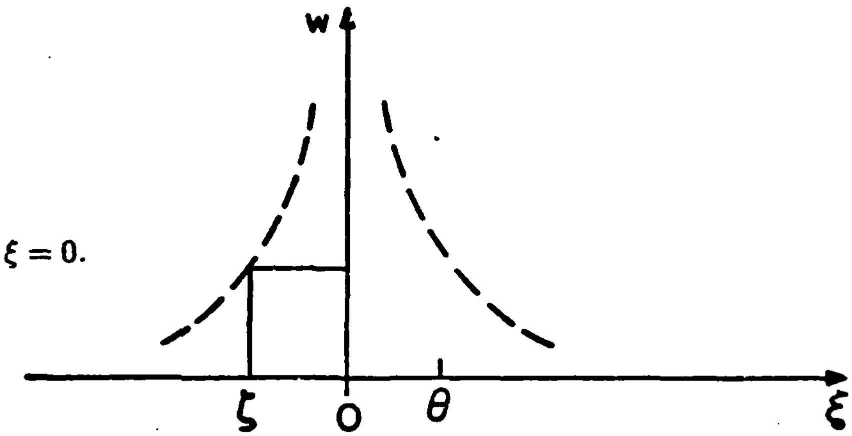
$$\lim_{\theta\to 0+} -p(w(\theta)) + \lim_{\zeta\to 0-} p(w(\zeta)) = \lim_{\substack{\theta\to 0+\\ \zeta\to 0-}} S(\zeta,\theta),$$

$$\lim_{\theta\to 0+} u(\theta) = \lim_{\zeta\to 0-} u(\zeta),$$

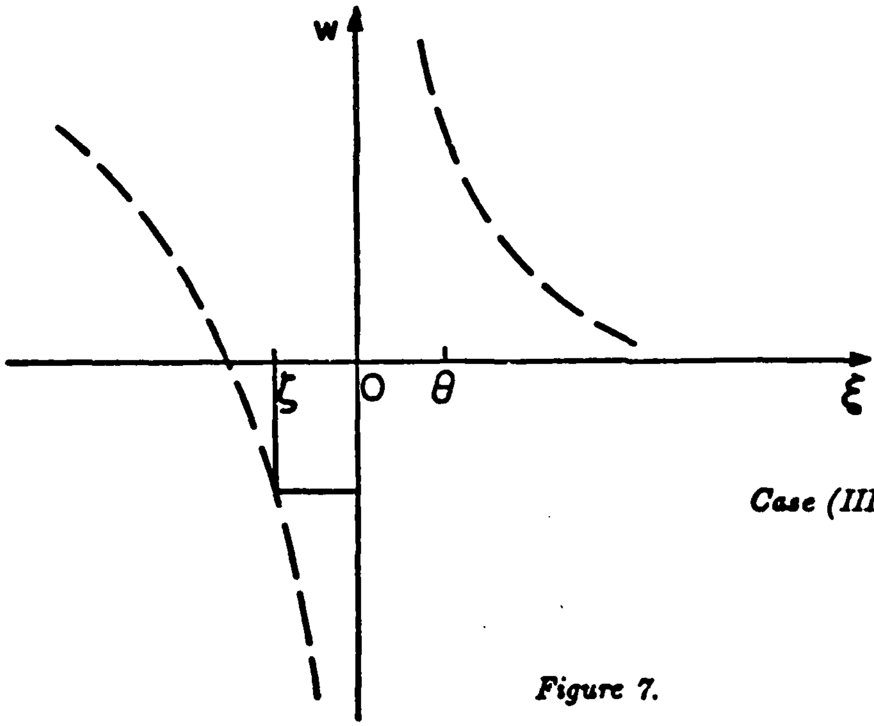
and (9.23_1) is always satisfied. Moreover if $\int_0^\xi u_\epsilon(\xi)d\xi$ is absolutely equicontinuous Vitali's theorem tells us we can pass the limit through the integral in (4.35) and hence show (4.23_2) holds as well. (In this we have of course $S(\zeta,\theta) = \lim\int_\zeta^\theta u(\xi)d\xi$.) Also the bounds on $S(\zeta,\theta)$ show that $p(\alpha) - p(\beta) \le -\lim_{\theta\to 0+} p(w(\theta)) + \lim_{\zeta\to 0-} p(w(\zeta)) \le 0$ holds in general.



Case (I): Infinite minimum at $\xi = 0$.



Case (II): Infinite maximum at $\xi = 0$.



Case (III): Infinite minimum-maximum

Figure 7.

Remark 4.12 As u_ϵ may have more than one critical point the argument used in [8] to show that the absolute equicontinuity of $\int_0^\xi u_\epsilon(\xi)d\xi$ is also *necessary*

to have (4.32_2) hold does not seem to apply.

Theorem 4.13 The functions $u(\xi)$, $w(\xi)$ defined by Lemma 4.5 provide a solution of the Riemann problem provided the pressure p equilibriates across the stagnant phase boundary at $\xi = 0$, i.e.

$$\lim_{\xi \to 0-} p(w(\xi)) = \lim_{\xi \to 0+} p(w(\xi)).$$

Proof. Use Lemma 4.11.

Acknowledgement

I would like to thank the Division of Applied Mathematics of Brown Univeristy and Departments of Theoretical and Applied Mathematics of the Weizmann Institute of Science for supporting me in part during the course of this work. Particular thanks goes to the Michael family which provided the funds for the Michael Professorship which I held at the Weizmann Institute. Personal thanks go to my hosts: at Brown, Profs. D.E. McClure and C.M. Dafermos; at the Weizmann Institute, Profs. L.A. Segel and Z. Artstein. Finally an extra note of appreciation is extended to Prof. Z. Artstein for his valuable suggestions and comments on the research presented here.

References

1. R.D. James, The propagation of phase boundaries in elastic bars, *Archive for Rational Mechanics and Analysis.*

2. M. Shearer, The Riemann problem for a class of conservation laws of mixed type, *J. Differential Equations* 46 (1982), 426–443.

3. M. Shearer, Nonuniqueness of admissible solutions of Riemann initial value problems for a system of conservation laws of mixed type, *Archive for Rational Mechanics and Analysis* 93 (1986), 45–59.

4. M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid, *Archive for Rational Mechanics and Analysis* 81 (1983), 301–315.

5. R. Hagan and M. Slemrod, The viscosity-capillarity criterion for shocks and phase transitions, *Archive for Rational Mechanics and Analysis* 83 (1984), 333–361.

6. M. Slemrod, Dynamics of first order phase transition, in *Phase Transformations and Material Instabilities in Solids*, ed. M. Gurtin, Academic Press: New York (1984), 163–203.

7. C.M. Dafermos, Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method, *Archive for Rational Mechanics and Analysis* 52 (1973), 1–9.

8. C.M. Dafermos, R.J. Di Perna, The Riemann problem for certain classes of hyperbolic systems of conservation laws, *J. Differential Equations* 20 (1976), 90–114.

9. C.S. Morawetz, On a weak solution for a transonic flow, *Comm. Pure and Applied Math.* 38 (1985), 797–818.

10. J. Mawhin, Topological degree methods in nonlinear boundary value problems, Conference Board of Mathematical Sciences Regional Conference Series in Mathematics, No. 40, American Mathematical Society (1979).

11. I.P. Natanson, Theory of functions of a real variable, Vol. 1, F. Ungar Publishing Co., New York (1955).

12. M. Shearer, Dynamic phase transitions in a van der Waals gas, to appear *Quarterly of Applied Math.*

13. A.S. Kalasnikov, Construction of generalized solutions of quasilinear equations of first order without convexity conditions as limits of solutions of parabolic equations with a small parameter, *Dokl. Akad. Nauk. SSR* 127 (1959), 27–30 (Russian).

14. V.A. Tupciev, The asymptotic behavior of the solution of the Cauchy problem for the equation $\epsilon^2 t u_{xx} = u_t + [\varphi(u)]_x$ that degenerates for $\xi = 0$ into the probl̲m of the decay of an arbitrary discontinuity for the case of a rarefaction wave. *Z. Vycisl. Mat. Fiz.* 12 (1972), 770–775; English translation in *USSR Comput. Math. and Phys.* 12.

15. V.A. Tupciev, On the method of introducing viscosity in the study of problems involving decay of a discontinuity, *Dokl. Akad. Nauk. SSR* 211 (1973), 55–58; English translation in *Soviet. Math. Dokl.* 14.

16. C.M. Dafermos, Structure of solutions of the Riemann problem for hyperbolic systems of conservation laws, *Arch. for Rational Mechanics and Analysis* 3 (1974), 203–217.

END
DATE
FILMED
6-1988
DTIC